Advanced Real Analysis Tutorial 04

Yu Junao

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Contents

1 Review

1.1 Compactness of L^p Spaces

Compactness is rather dearth in infinitely dimensional Banach spaces. Even L^p spaces, needs extra conditions to achieve compactness.

Theorem 1 (Riesz-Fréchet-Kolmogorov). *For $1 \leq p < +\infty$, a subset $\mathcal{F}$ of L^p is compact if and only if*

1. *$\mathcal{F}$ is bounded;*
2. *The L^p integration of $\mathcal{F}$ is uniformly absolutely continuous,*
$$\lim_{h \to \infty} \sup_{f \in \mathcal{F}} \|f(x+h) - f(x)\|_p = 0;$$
3. *The L^p integration of $\mathcal{F}$ uniformly vanishes at infinity,*
$$\lim_{R \to \infty} \sup_{f \in \mathcal{F}} \int_{B_R(0)^c} |f|^p = 0.$$

This theorem is a result of **Arzela-Ascoli theorem**, the most general compactness criterion.

However, it is usually not that practical to verify these properties. In possession of fewer properties, we might request weaker conclusions.

Theorem 2 (Eberlein-Smulian). *A bounded subset of L^p is weakly compact if $1 < p < +\infty$, namely there exists a weakly convergent subsequence.*

Remark 1. *This conclusion is actually applicable for general reflexive spaces.*

Boundedness is a simpler condition. In practice, we usually obtain a weak limit of a subsequence, and then utilize some magical tricks (such as compact embedding) to prove its strong convergence.

Since L^1 is not reflexive, more conditions are required to form a weakly compact subset.

Theorem 3 (Dunford-Pettis). *Consider the L^1 space on a σ-finite space. A subset $\mathcal{F}$ of it is compact if and only if*

1. *$\mathcal{F}$ is bounded;*
2. *$\mathcal{F}$ is uniformly absolutely continuous,*
$$\lim_{\mu(A) \to 0} \sup_{f \in \mathcal{F}} \int_A |f| \, \mathrm{d}\mu = 0;$$

3. $\mathcal{F}$ is equi-tight, namely $\forall\, \varepsilon > 0, \exists\, K \subset X, \mu(K) < +\infty$ such that

$$\sup_{f \in \mathcal{F}} \int_{K^c} |f| \, \mathrm{d}\mu < \varepsilon.$$

1.2 Radon Measures

For a measure space X that is not σ-finite, say just **locally compact Hausdorff** (LCF), it is pointless to require a high regularity. We extend such measures with a reduction of property.

Definition 1 (Radon measure)**.** *A Radon measure is a Borel measure that is finite on compact sets outer regular for Borel sets, and inner regular for open sets. Moreover, if it is inner regular for all Borel set, we call it a regular measure.*

Obviously, we extract principal properties from general concrete Borel measures to establish Radon set. It is a universal method of extending a concept.

Although the measure is worse, we can look for better dual objects to "reconcile" them.

Theorem 4 (Reisz representation theorem for $C_c(X)$)**.** *A linear function I is* ***positive*** *if $I(f) \geq 0$ for every $f \geq 0$. Given a positive linear functional on $C_c(X)$, there exists a unique Radon measure μ such that*

$$I(f) = \int_X f \, \mathrm{d}\mu.$$

Moreover, μ satisfies

$$\begin{aligned} \mu(U) &= \sup\{I(f) \mid f \in C_c(X), 0 \leq f \leq 1, \mathit{supp} f \subset U\}, \\ \mu(K) &= \inf\{I(f) \mid f \in C_c(X), f \geq \chi_K\} \end{aligned}$$

where U is open while K is compact.

In fact, Radon measures behave almost the same as Borel measures.

Theorem 5 (Properties of Radon measures)**.** *Let μ be a Radon measure on X, then*

1. *μ is regular if X is σ-finite;*

2. *$C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < +\infty$;*

3. ***Lusin's theorem*** *still holds;*

4. ***Tietze extension*** *is valid.*

Since $C_0(X)$ is the closure of $C_c(X)$ under maximum modulus norm, we can extend Riesz representation theorem to $C_0(X)$.

Theorem 6 (Reisz representation theorem for $C_0(X)$)**.** *Given a function $f \in C_0(X)$ and a complex Radon measure μ, the positive linear functional*

$$I_\mu(f) = \int_X f \, \mathrm{d}\mu.$$

is an isometry from $M(X)$ to $(C_0(X))^*$, where $M(X)$ is the collection of all complex Radon measures on X.

Remark 2. *The real and imaginary parts of a complex Radon measure are allowed to be signed measures.*

As is shown in our homework, $M(X)$ is a linear space under the **total variation norm**

$$\|\mu\| = |\mu|(X) = \mu_1(X) + \mu_2(X) + \nu_1(X) + \nu_2(X), \ \mu = \mu_1 - \mu_2 + i(\nu_1 - \nu_2).$$

Note the inclusion relation

$$C_c(X) \subset C_0(X) \subset C_b(X),$$

we can extend the concept "convergence" to measures through duality.

Definition 2 (Convergence modes of measures)**.** *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of Radon measures, then*

1. *μ_n converges to μ* ***vaguely*** *if*

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu, \ \forall f \in C_c(X);$$

2. *μ_n converges to μ* ***weakly*** *if*

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu, \ \forall f \in C_0(X);$$

3. *μ_n converges to μ* ***narrowly*** *if*

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu, \ \forall f \in C_b(X);$$

Former convergences are a litter weaker than latter ones.

2 Solutions to Homework

2.1 Exercise 6.2.22

(1)

Proof. As Riemann-Lebesgue lemma implies, if $f \in L^2(X)$, then

$$\lim_{n\to\infty}\int_0^1 f(x)\cos(2\pi nx)\,\mathrm{d}x = 0.$$

Therefore, $\{\cos(2\pi nx)\}_{n=1}^{\infty}$ converges weakly to 0.

For $\varepsilon \in (0,1)$, it is easy to check that

$$\mu\left\{|\cos(2\pi nx)| > \frac{1}{2}\right\} = \frac{2}{3},$$

independent of n.

Suppose that $\{\cos(2\pi nx)\}_{n=1}^{\infty}$ converges to 0 almost everywhere. By dominant convergence theorem, we have

$$0 = \int_0^1 \lim_{n\to\infty}\cos^2(2\pi nx)\,\mathrm{d}x = \frac{1}{2}\lim_{n\to\infty}\int_0^1(1+\cos(4\pi nx))\,\mathrm{d}x = \frac{1}{2},$$

a contradiction! □

(2)

Proof. Since f_n converges to 0 pointwise except for $x = 0$, we conclude that $f_n \to 0$ almost everywhere. For $\varepsilon > 0$, we have

$$\mu\{|f_n| > \varepsilon\} \le \frac{1}{n} \to 0,\ n\to\infty,$$

thus $f_n \to 0$ in measure.

It is obvious that $g = \chi_{[0,1]}$ belongs to L^p for all $p \in [1, +\infty]$, but

$$\int f_n g = \int f_n = 1 > 0.$$

Therefore, f_n never converges weakly in L^p. □

Remark 3. *It is almost impossible to directly show that $\{\cos(2\pi nx)\}_{n=1}^{\infty}$ does not converge almost everywhere.*

2.2 Exercise 6.5.41

Proof. Without loss of generality suppose $p < q$. T is a linear operator since

$$\begin{aligned}\int T(\lambda_1 f_1 + \lambda_2 f_2)g &= \int (\lambda_1 f_1 + \lambda_2 f_2)Tg \\ &= \lambda_1 \int f_1 Tg + \lambda_2 \int f_2 Tg \\ &= \int (\lambda_1 Tf_1 + \lambda_2 Tf_2)g, \ \forall\, g \in L^p \cap L^q.\end{aligned}$$

We first show that T is bounded on L^q. By the duality expression of L^q norm, we have for $f \in L^q$ that

$$\begin{aligned}\|Tf\|_q &= \sup\left\{ \int (Tf)g \,\middle|\, g \in L^p, \|g\|_p = 1 \right\} \\ &= \sup\left\{ \int f(Tg) \,\middle|\, g \in L^p, \|g\|_p = 1 \right\} \\ &\leq \sup\left\{ \|f\|_q \|Tg\|_p \mid g \in L^p, \|g\|_p = 1 \right\} \\ &\leq \sup\left\{ \|T\|_{p\to p} \|f\|_q \|g\|_p \mid g \in L^p, \|g\|_p = 1 \right\} \\ &= \|T\|_{p\to p} \|f\|_q.\end{aligned}$$

Riesz-Thorin interpolation theorem implies T is also bounded on L^r.

It is easy to verify that L^p is dense in L^r. Additionally, T is continuous since it is linear and bounded. As a result, the extension is unique. □

Remark 4. *A more complicated approach is the approximation of simple function, which is unique.*

2.3 Suppliments

Exercise 1 (Urysohn's lemma)**.** *Given an open $\Omega \subset \mathbb{R}^n$ and a compact set $K \subset \Omega$. Prove: there exists $\varphi \in C_c^\infty(\Omega)$ such that $\varphi|_K = 1$ and $0 \leq \varphi \leq 1$.*

Proof. Without loss of generality assume Ω is bounded. Otherwise, there exists $r > 0$ such that $K \subset B_r(0)$, thus we can substitute Ω with $\Omega \cap B_r(0)$ and let $\varphi = 0$ in $\Omega \backslash \overline{B_r(0)}$.

The boundedness of Ω admits the existence of a sufficiently large $R > 0$ such that $\Omega \subset B_R(0)$. It is easy to verify that $\overline{B_R(0)} \backslash \Omega$ is a bounded closed set, and then a compact set.

Now we are going to prove there is a positive distance between K and $\overline{B_R(0)} \backslash \Omega$. If

$$\inf\left\{ |x-y| \,\middle|\, x \in K, y \in \overline{B_R(0)} \backslash \Omega \right\} = 0,$$

then there is a sequence $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $|x_n - y_n| < \frac{1}{n}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, it possesses a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, converging to $x \in K$. Moreover, there is a convergent subsequence of $\{y_{n_k}\}_{k=1}^{\infty}$, converging to $y \in \overline{B_R(0)}\backslash\Omega$. As a result, we have

$$x = y \Longrightarrow \overline{B_R(0)}\backslash\Omega \cap K \neq \varnothing \Longrightarrow \Omega^c \cap K \neq \varnothing \Longrightarrow \Omega^c \cap \Omega \neq \varnothing,$$

a contradiction.

Back to the point, take the mollifier η_ε such that $\varepsilon < \frac{d}{2}$, and define

$$\varphi(x) = \int \chi_{\tilde{K}}(y)\eta_\varepsilon(x-y)\,\mathrm{d}y \in C_c^\infty(\Omega).$$

where $K \subset \tilde{K} \subset \Omega$ satisfies

$$\tilde{K} = \left\{ x \in \Omega \,\middle|\, \inf_{y \in K} |x-y| \leq \frac{d}{2} \right\}.$$

This φ definitely satisfies our requirements since

$$\begin{aligned} x \in K &\Longrightarrow \varphi(x) = \int_{\tilde{K}} \eta_\varepsilon(x-y)\,\mathrm{d}y = 1, \\ x \in \Omega^c &\Longrightarrow \varphi(x) = \int 0\,\mathrm{d}y = 0. \end{aligned}$$

□

Remark 5. *Urysohn's lemma is still correct in some general topological spaces, yet the proof will be much more tricky. Some student took $\varepsilon = d$, unfortunately, this mollifier is to rough to satisfy our requirements.*

Exercise 2. *Show that finite measure space $M(X, B_X, \mathbb{C})$ is a Banach space under total variation norm.*

Proof. Let $\{\mu_n\}_{n=1}^{\infty} \subset M(X, B_X, \mathbb{R})$ be a sequence of complex Borel measures such that

$$\lim_{m,n\to\infty} \|\mu_m - \mu_n\| = \lim_{m,n\to\infty} |\mu_m - \mu_n|(X) = 0.$$

For every $E \in B_X$, we have

$$|\mu_m(E) - \mu_n(E)| \leq |\mu_m - \mu_n|(E) \leq |\mu_m - \mu_n|(E) \to 0,\ m, n \to \infty.$$

That is to say $\{\mu_n(E)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Therefore, we can define the limit set function "pointwise" by

$$\mu(E) = \lim_{n\to\infty} \mu_n(E).$$

Finally, we need to verify that μ is a Borel measure in deed. Actually, it is obvious that

$$\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = 0.$$

Moreover, let $\{E_k\}_{k=1}^\infty$ be a disjoint sequence of Borel measurable sets, then

$$\mu\left(\bigsqcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} \mu_n\left(\bigsqcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} \sum_{k=1}^\infty \mu_n(E_k) = \sum_{k=1}^\infty \mu(E_k).$$

Here it is reasonable to swap two limits since the finiteness of μ_n implies the series is absolutely convergent.

We have proved that μ is a measure, and by definition Borel sets are μ-measurable and μ is finite. As a result, we conclude that μ is a finite Borel measure and thus $M(X, B_X, \mathbb{C})$ is a Banach space. □

Remark 6. *There is another ingenious approach provided by a couple of students by considering*

$$\rho = \sum_{n=1}^\infty \frac{1}{2^n}|\mu_n|$$

for a Cauchy sequence $\{\mu_n\}_{n=1}^\infty$. In this sense, μ_n is absolutely continuous with respect to λ. Therefore, the convergence of measure is converted to that of a sequence of L^1 functions, which is more familiar to us.

Exercise 3. *If X is a Banach space, then F is precompact if and only if $\forall\, \varepsilon > 0$, there exists a precompact K_ε such that*

$$F \subset K_\varepsilon + B_\varepsilon(0) = \{f + g \mid f \in K_\varepsilon, g \in B_\varepsilon(0)\}.$$

Proof. $\Longrightarrow$:

Take $K_\varepsilon = F$.

$\Longleftarrow$:

If F is not precompact, then there exist a sequence $\{\varphi_n\}_{n=1}^\infty \subset F$ such that every subsequence of it is not Cauchy.

Consider decomposition $\varphi_n = f_n + g_n$ where $f_n \in K_\varepsilon, g_n \in B_\varepsilon(0)$ for a pending $\varepsilon > 0$, then

$$\|\varphi_m - \varphi_n\| \leq \|f_m - f_n\| + \|g_m - g_n\| \leq \|f_m - f_n\| + 2\varepsilon.$$

Since K_ε is precompact, there is a sufficient large N and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$\|f_{n_j} - f_{n_k}\| < \varepsilon, \ \forall\, n_j, n_k > N.$$

However, our assumption implies $\{\varphi_{n_k}\}_{n=1}^{\infty}$ is not Cauchy, hence there exist $\varepsilon_0 \geq 0$ and $j, k > N$ such that

$$\|\varphi_{n_j} - \varphi_{n_k}\| \geq \varepsilon_0.$$

Take $\varepsilon = \frac{\varepsilon_0}{3}$ and we obtain

$$\varepsilon_0 \leq \|\varphi_{n_j} - \varphi_{n_k}\| \leq \|f_{n_j} - f_{n_k}\| + 2\varepsilon < 3\varepsilon = \varepsilon_0,$$

a contradiction. □

Remark 7. *We can also construct a finite $\frac{\varepsilon}{2}$-net of $K_{\frac{\varepsilon}{2}}$, and then the fact*

$$B_{\frac{\varepsilon}{2}}(x) + B_{\frac{\varepsilon}{2}}(0) \subset B_\varepsilon(x)$$

implies the existence of an ε-net of F.

3 Rearrangement

Most measurable functions behave so terribly that we can hardly expect any remarkable properties from them. However, we can adjust the level set of a measurable function in order to make it "looks more regular".

This operation is somehow compatible with Lebesgue integration with focuses more on the range instead of domain. As we know, a lot of practical inequalities achieve the equality if the function is **symmetric**. That is our primary inspiration of rearrangement.

3.1 Schwartz Rearrangement

Symmetry is a concept concerned with metric, and we shall talk about rearrangement only on $\mathbb{R}^n$. In fact, we want to reshape an arbitrary irregular function into a **radial** one.

Definition 3 (Rearrangement of a bounded region)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded region, then the rearrangement of Ω is a ball $\Omega^* = B_R(0)$ such that $|\Omega^*| = |\Omega|$.*

Remark 8. *It is worthwhile to mention that some eigenvalue problems achieve their extrema if and only in $\Omega = \Omega^*$.*

In that sense, we can naturally symmetrize a characteristic function by

$$\chi_\Omega^* = \chi_{\Omega^*}.$$

The fundamental idea is to convert each lever set to a radially symmetric set centered at origin to construct a new function that decrease with respect to r. However, not every function is suitable for rearrangement. For example, we cannot rearrange $f(x) = x$ since it is impossible to determine its value at origin.

Definition 4 (Rearrangement of a function)**.** *Let $u(x)$ be a measurable function that vanishes at infinity, namely*

$$\lim_{|x|\to\infty} f(x) = 0,$$

*then the **Schwartz Rearrangement** of u is defined by*

$$u^*(x) = \int_0^{+\infty} \chi^*_{\{|u|>t\}}(x)\,\mathrm{d}t.$$

In other words, u^* is radially symmetric and decreasing function such that

$$|\{|u| > t\}| = |\{|u^*| > t\}|, \ \forall t \geq 0.$$

Let's calculate an example to deep our comprehension. Abstract

$$u(x) = \begin{cases} 0, & x \in (-\infty, 0) \cup (2, +\infty) \\ \frac{1}{2}x, & x \in [0, 2], \end{cases}$$

then

$$
\begin{aligned}
u^*(x) &= \int_0^{+\infty} \chi^*_{\{|u|>t\}}(x)\,\mathrm{d}t \\
&= \int_0^{+\infty} \chi_{\{|u|>t\}^*}(x)\,\mathrm{d}t \\
&= \int_0^{+\infty} \chi_{(t-1,1-t)}(x)\,\mathrm{d}t \\
&= \int_0^{+\infty} \chi_{\{0<t<1+x\}\cap\{0<t<1-x\}}(t)\,\mathrm{d}t \\
&= \max\{0, \min\{1+x, 1-x\}\} \\
&= \begin{cases} 1+x, & x \in [-1,0], \\ 1-x, & x \in (0,1], \\ 0, & x \in (-\infty,-1) \cup (1,+\infty). \end{cases}
\end{aligned}
$$

Let $v(r) = u^*(|x|)$, then

$$
v(r) = \begin{cases} 1-r, & 0 \le r \le 1, \\ 0, & r > 1. \end{cases}
$$

The rearranged function somehow maintains a couple of properties of the original function.

Theorem 7. *Rearrangement is L^p norm invariant, namely*

$$
\|u\|_p = \|u^*\|_p.
$$

Proof. The case $p = \infty$ is trivial. Assume $1 \le p < +\infty$, the Layer cake representation implies

$$
\begin{aligned}
\|u\|_p^p &= \int_0^{+\infty} p\lambda^{p-1} |\{|u| > \lambda\}|\,\mathrm{d}\lambda \\
&= \int_0^{+\infty} p\lambda^{p-1} |\{|u^*| > \lambda\}|\,\mathrm{d}\lambda \\
&= \|u^*\|_p^p
\end{aligned}
$$

□

3.2 Pólya-Szegö inequality

In this section, we shall show a remarkable result on the L^p norm of gradient of the rearranged function.

Theorem 8 (Coarea formula)**.** *For $u \in C^1(\mathbb{R}^n), v \in L^1(\mathbb{R}^n)$, we have*

$$\int_\Omega v|\nabla u| \,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\{u=\lambda\}\cap\Omega} v \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}\lambda.$$

This theorem is quite useful yet its proof is sophisticated. The most general coarea formula is written in form of Riemannian geometry. We omit the proof and interested students please refer to Evans' textbook on geometric measure theory or any lecture note on Riemannian geometry.

In comparison of its original statement, we rely more on its corollary in Euclidean spaces.

Theorem 9. *For a function $u \in C^1(\mathbb{R}^n)$, define*

$$\mu(t) = |\{|u| > t\}| = \int \chi_{\{|u|>t\}},$$

then we have

$$\mu'(t) = -\int_{\{u=t\}} \frac{1}{|\nabla u|} \,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. Let $v = \frac{1}{|\nabla u|}$ and $\Omega_t = \{|u| > t\}$ then

$$\mu(t) = |\Omega_t| = \int_t^{+\infty} \left(\int_{\{u=\lambda\}} \frac{1}{|\nabla u|} \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}\lambda.$$

Construct the quotient of differences

$$\frac{\mu(t+h) - \mu(t)}{h} = -\frac{1}{h} \left(\int_t^{t+h} \left(\int_{\{u=\lambda\}} \frac{1}{|\nabla u|} \,\mathrm{d}\mathcal{H}^{n-1} \right) \mathrm{d}\lambda \right).$$

Let h tends to 0, then

$$\mu'(t) = -\int_{\{u=t\}} \frac{1}{|\nabla u|} \,\mathrm{d}\mathcal{H}^{n-1}.$$

□

Remark 9. *With the assistance of μ, we can present Schwartz rearrangement in a different manner*

$$v(r) = \sup \left\{ \lambda \,\middle|\, \mu(\lambda) > \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} |x|^n \right\},$$

where $v(|x|) = u^*(x)$

This conclusion does not involve rearrangement, but we shall take advantages of it in next proof.

As we have known since the second grade of elementary school, a planar shape of given area enjoys the minimum perimeter if and only if it is a circle. We can present this fact in higher dimensions with the assistance of rearrangement.

Theorem 10 (Isoperimetric inequality for level sets)**.** *For $u \in C_0^1(\mathbb{R}^n)$, we have*

$$\mathcal{H}^{n-1}(\{u^* = t\}) \leq \mathcal{H}^{n-1}(\{u = t\}).$$

Proof. Due to the fact that

$$|\{u^* > t\}| = |\{u > t\}|,$$

the two set $\{u^* = t\}$ and $\{u = t\}$ are respectively the boundaries of two regions equal in $\mathcal{H}^n$.

By definition, $\{u^* > t\}$ is a ball, hence classical isoperimetric inequality implies

$$\mathcal{H}^{n-1}(\{u^* = t\}) \leq \mathcal{H}^{n-1}(\{u = t\}).$$

□

Ultimately, we have converted a nontrivial inequality into a direct corollary.

Theorem 11 (Pólya-Szegö Inequality)**.** *Let $u \in C_0^1(\mathbb{R}^n), \nabla u \in L^p$, then for $1 \leq p \leq +\infty$ we have*

$$\|\nabla u^*\|_p \leq \|\nabla u\|_p.$$

Proof. Hölder's inequality implies

$$\begin{aligned}
\mathcal{H}^{n-1}(\{u = t\})^p &= \left(\int_{\{u=t\}} \mathrm{d}\mathcal{H}^{n-1}\right) \\
&= \left(\int_{\{u=t\}} |\nabla u|^{\frac{p-1}{p}} |\nabla u|^{-\frac{p-1}{p}} \, \mathrm{d}\mathcal{H}^{n-1}\right)^p \\
&\leq \left(\int_{\{u=t\}} |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}\right) \left(\int_{\{u=t\}} \frac{1}{|\nabla u|} \, \mathrm{d}\mathcal{H}^{n-1}\right)^{p-1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathcal{H}^{n-1}(\{u^* = t\})^p &= \left(\int_{\{u^*=t\}} \mathrm{d}\mathcal{H}^{n-1}\right) \\
&= \left(\int_{\{u^*=t\}} |\nabla u^*|^{\frac{p-1}{p}} |\nabla u^*|^{-\frac{p-1}{p}} \, \mathrm{d}\mathcal{H}^{n-1}\right)^p \\
&= \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}\right) \left(\int_{\{u^*=t\}} \frac{1}{|\nabla u^*|} \, \mathrm{d}\mathcal{H}^{n-1}\right)^{p-1}
\end{aligned}$$

since the radial symmetry of u^* leads to the equal sign.

By isoperimetric inequality, we have

$$\mathcal{H}^{n-1}(\{u^* = t\})^p \leq \mathcal{H}^{n-1}(\{u = t\})^p,$$

thus

$$\int_0^M \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1} \right) \leq \int_0^M \left(\int_{\{u=t\}} |\nabla u|^{p-1} \right).$$

The case $p = \infty$ is obvious, otherwise there is a sufficient large M such that

$$\begin{aligned} \int |\nabla u^*|^p &= \int_0^M \left(\int_{\{u^*=t\}} |\nabla u^*|^{p-1} \right) \\ &\leq \int_0^M \left(\int_{\{u=t\}} |\nabla u|^{p-1} \right) \\ &= \int |\nabla u|^p. \end{aligned}$$

□

3.3 Sharp Sobolev Inequality

As one of the most significant L^p inequality in partial differential equations, Sobolev inequality unveils the relation between a function and its gradient.

Theorem 12 ((Talenti, 1993))**.** *Fix n and $1 \leq p < n$, and let u be a sufficiently smooth function that vanishes at infinity. The Sobolev inequality*

$$\|u\|_{p^*} \leq C\|\nabla u\|_p$$

holds for

$$C = \frac{1}{\sqrt{\pi} n^{\frac{1}{p}}} \left(\frac{p-1}{n-p} \right)^{1-\frac{1}{p}} \left(\frac{\Gamma(1+\frac{n}{2})\Gamma(n)}{\Gamma(\frac{n}{p})\Gamma(1+n-\frac{n}{p})} \right),$$

and the equality is achieved if and only if

$$u = \frac{1}{(a + b|x|^{\frac{p}{p-1}})^{1-\frac{n}{p}}}, \ a, b > 0.$$

Remark 10. *We can prove Sobolev inequality with only primary calculus and Hölder's inequality, but such an approach fails to calculate the best constant C.*

Proof. As is proved, for $p \geq 1$ we have

1. $\|u^*\|_p = \|u\|_p$,
2. $\|\nabla u^*\|_p \leq \|\nabla u\|_p$.

Therefore, we only need to prove the inequality for redial function $v(r) = v(|x|) = u^*(x)$. The norms are simplified to

$$\|u^*\|_{p^*} = \left(\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^{+\infty} |v(r)|^{p^*} r^{n-1} \,\mathrm{d}r\right)^{\frac{1}{p^*}},$$
$$\|\nabla u^*\|_p = \left(\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^{+\infty} |v'(r)|^p r^{n-1} \,\mathrm{d}r\right)^{\frac{1}{p}}.$$

Let $\varphi \in C_c^\infty(\mathbb{R})$ be a non-negative function, we shall apply the calculus of variations as a result of the density of C_c^∞. Define

$$I[v] = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})^{\frac{p}{p^*}-1} \frac{\|\nabla u^*\|_p^p}{\|u^*\|_{p^*}^p} = \frac{\int_0^{+\infty} |v'(r)|^p r^{n-1} \,\mathrm{d}r}{\left(\int_0^{+\infty} |v(r)|^{p^*} r^{n-1} \,\mathrm{d}r\right)^{\frac{p^*}{p}}},$$

and let

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} I[v + \varepsilon\varphi] = 0$$

for non-constant $v \in C^1(\mathbb{R})$.

Since the positiveness of denominator, we only compute the numerator.

$$\begin{aligned} 0 = &\left(\int_0^{+\infty} p|v' + \varepsilon\varphi'|^{p-2} (v' + \varepsilon\varphi') \varphi' r^{n-1} \,\mathrm{d}r\right) \left(\int_0^{+\infty} |v + \varepsilon\varphi| \,\mathrm{d}r\right)^{\frac{p}{p*}} \\ &- \frac{p}{p^*}\left(\int_0^{+\infty} |v' + \varepsilon\varphi'|^p r^{n-1} \,\mathrm{d}r\right) \left(\int_0^{+\infty} |v + \varepsilon\varphi|^{p*} r^{n-1} \,\mathrm{d}r\right)^{\frac{p}{p^*}-1} \\ &\cdot \left(\int_0^{+\infty} p^*|v + \varepsilon\varphi|^{p^*-2}(u + \varepsilon\varphi)\varphi r^{n-1} \,\mathrm{d}r\right). \end{aligned}$$

Take $\varepsilon = 0$, and

$$\begin{aligned} 0 = &\left(\int_0^{+\infty} p|v'|^{p-2} v' \varphi' r^{n-1} \,\mathrm{d}r\right) \left(\int_0^{+\infty} |v| \,\mathrm{d}r\right)^{\frac{p}{p*}} \\ &- \frac{p}{p^*}\left(\int_0^{+\infty} |v'|^p r^{n-1} \,\mathrm{d}r\right) \left(\int_0^{+\infty} |v|^{p*} r^{n-1} \,\mathrm{d}r\right)^{\frac{p}{p^*}-1} \left(\int_0^{+\infty} p^*|v|^{p^*-2} u\varphi r^{n-1} \,\mathrm{d}r\right). \end{aligned}$$

Equivalently, we have

$$\int_0^{+\infty} |v'|^{p-2}v'\varphi' r^{n-1}\,\mathrm{d}r = \left(\frac{\int_0^{+\infty} |v'|^p r^{n-1}\,\mathrm{d}r}{\int_0^{+\infty} |v|^{p^*} r^{n-1}\,\mathrm{d}r}\right)\int_0^{+\infty} |v|^{p^*-2}v\varphi r^{n-1}\,\mathrm{d}r.$$

Integrating by parts, the left hand side is converted to

$$\int_0^{+\infty} |v'|^{p-2}v'\varphi' r^{n-1}\,\mathrm{d}r = -\int_0^{+\infty} \left(|v'|^{p-2}v' r^{n-1}\right)' \varphi\,\mathrm{d}r.$$

Since the critical point of $I[v]$ is independent of the choice of φ, we have

$$-\left(|v'|^{p-2}v' r^{n-1}\right)' = C|v|^{p^*-2}v r^{n-1}.$$

The solutions to this ordinary differential equations are in form of

$$v(r) = \frac{1}{\left(a + br^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}},\ \ a, b > 0,$$

namely

$$u(x) = \frac{1}{\left(a + b|x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}},\ \ a, b > 0.$$

Returning to the original inequality, we obtain the best constant

$$C = \frac{1}{\sqrt{\pi} n^{\frac{1}{p}}}\left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}}\left(\frac{\Gamma(1+\frac{n}{2})\Gamma(n)}{\Gamma(\frac{n}{p})\Gamma(1+n-\frac{n}{p})}\right).$$

□

There are a lot of inequalities developed from Sobolev inequality, and this constant sometimes plays an indispensable part in some problems such as **Yamabe's problem**.