Advanced Real Analysis Tutorial 05

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Contents

1 Review

1.1 Pointwise Convergence of Fourier Series

Fourier series is the most classical topic on Fourier analysis. Taylor expansion makes it possible to approximate elementary functions with polynomials, one of the simplest variety of functions that admit quantities of practical properties.

Among all kinds of waves, simple harmonic waves are the most common. Therefore, it is natural to ask whether all strange waveforms with a real physical background could be approximated by simple harmonic waves, namely sin and cos. This is actually the origin of **Fourier series**.

Unfortunately, the answer is "not necessary". As a result, we need to study under what condition the Fourier series of a function converges or additionally converge to the function itself. To calculate the partial sum, we usually apply the exponential expression according to Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Since trigonometric functions on $\mathbb{R}$ are periodic, we only need to consider

$$\mathbb{T} = [0,1]/\sim$$

where $0 \sim 1$. As we learned in functional analysis,

$$\left\{e^{2n\pi ix} \mid n \in \mathbb{Z}\right\}$$

is an orthogonal basis of in the sense of L^2 inner product, thus the **Fourier coefficients** of f are defined as

$$\hat{f}(n) = \int_{\mathbb{T}} f(x)e^{2n\pi ix}\,\mathrm{d}x.$$

Now we can introduce the definition of Fourier series.

Definition 1 (Fourier series)**.** *The **Fourier series associated with** f is*

$$f(x) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{2n\pi ix}.$$

Remark 1. *L^1 functions are preferred here since*

$$|\hat{f}(n)| = \left|\int_{\mathbb{T}} f(x)e^{2n\pi ix}\,\mathrm{d}x\right| \le \int_{\mathbb{T}} |f(x)||e^{2n\pi ix}|\,\mathrm{d}x \le \|f\|_1,$$

which implies the Fourier coefficients are well defined. It is worth mentioning that $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ as a result of Hölder's inequality.

To study its convergence, we need to start from the **partial sum**

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2k\pi i x} = (f * D_N)(x)$$

where D_N is the N-th **Dirichlet kernel**

$$D_N(x) = \sum_{n=-N}^{N} e^{2n\pi i x} = \frac{\sin(2N+1)\pi x}{\sin \pi x}.$$

Dirichlet kernel does not possess perfect property as it jumps up and down, yet it still possesses a few properties.

Theorem 1 (Properties of Dirichlet kernel)**.** *Let D_N be the Dirichlet kernel, then*

1. $\int_{\mathbb{T}} D_N = 1$;
2. $|D_N(x)| \leq C \min\{N, x^{-1}\}$;
3. $c \log N \leq \|D_n\|_1 \leq C \log N$.

Next two theorems are extended from that introduced in mathematical analysis.

Theorem 2 (Riemann-Lebesgue)**.** *High frequency terms tend to vanish, namely*

$$f \in L^1(\mathbb{T}) \Longrightarrow \lim_{n\to\infty} \hat{f}(n) = 0.$$

It implies that the Fourier coefficients of an L^1 function belong to $C_0(\mathbb{Z})$.

Remark 2. *This is still correct for $f \in L^2$ as a corollary of Parseval identity.*

Theorem 3 (Riemann localization theorem)**.** *If f vanishes in a neighborhood of x, then*

$$\lim_{N\to 0} S_N f(x) = 0.$$

Well known to analysts, it is easier to verify whether something tends to 0 rather than a concrete nonzero value. With the assistance of the two theorems concerned with convergence above, we introduce a classic result as follow.

Theorem 4 (Dini's criterion)**.** *If for some $x \in \mathbb{T}$, there exists $\delta > 0$ such that*

$$\int_{|t|<\delta} \left| \frac{f(x-t) - f(x)}{t} \right| \mathrm{d}t < +\infty,$$

then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

A direct corollary is $f \in C^\alpha(\mathbb{T})$ implies that the Fourier series associate with f converges pointwise to f itself.

Remark 3. *Dini's condition is not far away from the equivalent condition of the convergence of Fourier series, which has not been discovered.*

Dini's condition cannot represent a specific class of function and we need to verify it pointwise for a given function and sometimes fails to take effect. For example, square wave $f = \chi_{[\frac{1}{2},1]}$ is a common wave function, and Dini's condition fails at the discontinuous point $x = \frac{1}{2}$. Luckily, we have an more universal criterion which is still effective for some discontinuous functions.

Theorem 5 (Jordan's criterion)**.** *If f is of bounded variation in a neighborhood of x, then*

$$\lim_{N\to\infty} S_N f(x) = \frac{f(x^+) + f(x^-)}{2}.$$

1.2 Fejér kernel

To obtain the L^p convergence of Fourier series, we need something more regular than Dirichlet kernels. Since Cesàro mean enjoys better convergence convergence properties, we introduce another kernel.

Definition 2 (Fej'er kernel)**.** *Define Fej'er kernel as the Cesàro mean of Dirichlet kernel, namely*

$$F_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x) = \frac{\sin^2 (N+1)\pi x}{(N+1)\sin^2 \pi x}.$$

Similar with Dirichlet kernel, we have

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_N f(x) = (f * F_N)(x).$$

The Fejér kernels are a family of **approximate identity** which satisfies

Theorem 6 (Properties of Fejér kernel)**.** *A family of Fejér kernels has following properties*

1. *Boundedness:* $F_N \in L^\infty(\mathbb{T})$*;*
2. *Normalization:* $F_N \geq 0$ *and* $\int_{\mathbb{T}} F_N = 1$*;*
3. *Concentration:* $\int_{|x|>\delta} F_N(x)\,\mathrm{d}x \to 0$ *for any* $\delta > 0$ *as* $N \to \infty$*.*

The last property of approximate identity enables one to split domain of integration into a small neighborhood of origin and a large region distant from origin, which leads to the following results.

Theorem 7 (L^p convergence of Fourier series)**.** *Let* $f \in X$ *where* $X = C(\mathbb{T})$ *or* $L^p(\mathbb{T})$ *for* $1 \leq p < +\infty$*, then*

$$\lim_N \|f * F_N - f\|_X = 0,$$

which also implies that trigonometric polynomials are dense in such a functions space X*. An important corollary is the* ***Parseval identity***

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

for $f \in L^2(\mathbb{T})$*.*

Everything seems to have progressed smoothly for Fejér kernels thanks to the fine properties of approximate identity. When it comes back to Dirichlet kernels, we need more powerful conditions to maintain the convergence.

Theorem 8 (Convergence involving Dirichlet kernel)**.** *Let* X *be the function space defined in last theorem, then* $S_N f$ *converges to* f *in* X *if and only if*

$$\sup_N \|S_N\|_{X\to X} < +\infty.$$

1.3 Properties of Fourier Coefficients

Not all sequences in $C_0(\mathbb{Z})$ are the Fourier coefficients of a function. Moreover, sometimes the special properties of Fourier coefficients reflect that of the original function.

Theorem 9 (Bernstein's inequality)**.** *If $f \in L^p(\mathbb{T})$ satisfies $\hat{f}(n) = 0$ where $|n| > N$ for some N, then*

$$\|f'\|_p \leq CN\|f\|_p.$$

In that case, a L^p function on torus is differentiable if its Fourier coefficients belong to $C_c(\mathbb{Z})$. A dual result is

Theorem 10 (Reverse Bernstein's inequality)**.** *If $f \in L^p(\mathbb{T})$ and $f' \in L^p(\mathbb{T})$ then $\hat{f}(n) = 0$ for every $|n| < N$ implies*

$$\|f'\|_p \geq Cn\|f\|_p.$$

Remark 4. *Through boostrapping, such conclusions could be extended to higher derivatives.*

Next we introduce some necessary or sufficient conditions under which a sequence in $C_0(\mathbb{Z})$ is the Fourier coefficients of a function.

Theorem 11 (Symmetric case)**.** *For $\{a_n\}_{n\in\mathbb{Z}} \subset C_0(\mathbb{Z})$, it is the Fourier coefficients of a non-negative L^1 function if*

$$a_n = a_{-n}, \quad a_{n+1} + a_{n-1} - 2a_n \geq 0.$$

Theorem 12 (Antisymmetric case)**.** *If $\{a_n\}_{n\in\mathbb{Z}} \subset C_0(\mathbb{Z})$ such that $a_n = -a_{-n}$ is the Fourier coefficients of $f \in L^1(\mathbb{T})$, then*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < +\infty.$$

Intuitively, a regular function admits regular Fourier coefficients.

Theorem 13 (Regularity of Fourier coefficients)**.** *The regularity of Fourier coefficients is given as follow.*

1.

$$f \in AC(\mathbb{T}) \Longrightarrow \hat{f}(n) \leq \frac{C}{n};$$

2.

$$f \in C^k(\mathbb{T}), f^{(k)} \in AC(\mathbb{T}) \Longrightarrow |\hat{f}(n)| \leq \min_{0\leq j\leq k} \frac{\|f^{(j)}\|_1}{(2\pi|n|^j)}$$

3.

$$f \in C^\infty(\mathbb{T}) \Longrightarrow |\hat{f}(n)| \leq \min_{k\geq 0} \frac{\|f^{(k)}\|_1}{(2\pi|n|^k)}$$

4.

$$f \in C^\alpha(\mathbb{T}) \Longrightarrow |\hat{f}(n)| \leq \frac{C\|f\|_{C^\alpha}}{n^\alpha}.$$

Particularly for analytic function, we have a more elegant result.

Theorem 14. *f is analytic if and only if there exist $K > 0$ and $a > 0$ such that*

$$|\hat{f}(n)| \leq Ke^{-a|n|}.$$

1.4 Fourier Coefficients of a Borel Measure

Recall that we defined the convergence of Borel measures by duality. The same idea is accessible for Fourier Coefficients.

Definition 3 (Fourier coefficients of μ)**.** *Let $\mathcal{M}(\mathbb{T})$ be the collection of Borel measures on torus, then*

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-2n\pi ix} \, \mathrm{d}\mu(x).$$

It enjoys most properties of the Fourier coefficients of a function.

Theorem 15 (Absolute continuity)**.** *If $\mathrm{d}\mu = f \, \mathrm{d}x$ for some $f \in L^1(\mathbb{T})$, then*

$$\hat{\mu}(x) = \hat{f}(n).$$

Theorem 16 (Young's inequality)**.** *For $\mu \in \mathcal{M}(\mathbb{T})$ and $f \in C(\mathbb{T})$, we have*

$$\|f * \mu\|_p \leq \|\mu\|\|f\|_p,$$

where

$$(f * \mu)(x) = \int_{\mathbb{T}} f(x-y) \, \mathrm{d}\mu(y).$$

Theorem 17 (Weak * convergence)**.** *Let $\{\varphi_n\}_{n=1}^\infty$ be a family of approximate identity, then*

$$\varphi_n * \mu \to \mu$$

*in the sense of weak * as $n \to \infty$.*

Theorem 18 (Parseval identity for measures)**.** *Let $f \in C(\mathbb{T})$ and $\mu \in \mathcal{M}(\mathbb{T})$, then*

$$\int_{\mathbb{T}} \bar{f}\,\mathrm{d}\mu = \langle f, \mu \rangle = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \overline{\hat{f}(n)} \hat{\mu}(n).$$

A direct corollary is that $\hat{\mu}(n) = 0$ for every n implies $\mu = 0$.

Similarly, there is a criterion to identify whether a sequence is the Fourier coefficients of a measure.

Theorem 19. *For a complex sequence $\{a_n\}_{n\in\mathbb{Z}}$, there exists $\mu \in \mathcal{M}(\mathbb{T})$ such that $\hat{\mu}(n) = a_n$ if and only if*

$$\left| \sum_{n=-\infty}^{+\infty} \overline{\hat{P}(n)} a_n \right| \leq C \|P\|_{C(\mathbb{T})}$$

for some constant C independent of the selection of trigonometric polynomial P.

2 Solutions to Homework

2.1 Exercise 7.2.9

(1)

Proof. For an open set U, we have

$$\begin{aligned}
\nu'(U) &= \sup\left\{ \int f\varphi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \leq f \leq 1, \operatorname{supp} f \in U \right\} \\
&= \sup\left\{ \int \psi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \leq \psi \leq \varphi, \operatorname{supp} \psi \in U \right\} \\
&= \sup\left\{ \int_U \psi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \leq \psi \leq \varphi \right\} \\
&= \int_U \varphi\,\mathrm{d}\mu \\
&= \nu(U).
\end{aligned}$$

□

(2)

Proof. Since φ is continuous, the sets

$$V_k = \varphi^{-1}\left(2^{k-1}, 2^{k+1}\right), \ k \in \mathbb{Z}.$$

are open. Moreover, φ is positive, thus

$$X = \bigcup_{k=-\infty}^{+\infty} V_k.$$

Let E be a Borel set and define $E_k = E \cap V_k$. Note that φ is bounded in V_k, thus ν is absolutely continuous in respect to μ. Fix $\varepsilon > 0$, there exists $\delta_k > 0$ such that

$$\mu(U_k \backslash E_k) < \delta_k \Longrightarrow \nu(U_k \backslash E_k) < \frac{\varepsilon}{3 \cdot 2^{-|k|}}.$$

The outer regularity of μ implies the existence of an open U_k including E_k such that $\mu(U_k \backslash E_k) < \delta_k$.

Consider the open set

$$U = \bigcup_{k=-\infty}^{+\infty} U_k \supset \bigcup_{k=-\infty}^{+\infty} E_k \supset E,$$

we obtain

$$\nu(U \backslash E) \leq \sum_{k=-\infty}^{+\infty} \nu(U_k \backslash E_k) \leq \sum_{k=-\infty}^{+\infty} \frac{\varepsilon}{3 \cdot 2^{-|k|}} = \varepsilon,$$

namely the outer regularity of ν. □

Remark 5. *The preimage $\varphi^{-1}[2^{k-1}, 2^{k+1}]$ is not necessarily compact since φ^{-1} is not necessarily continuous. Therefore, all compactness arguments are incorrect.*

The solution did not reflect entire properties of the binary decomposition. As is shown in a significant proportion of submitted homework, we can utilize the bound

$$\varphi(x) < 2^{k+1}, \ \forall\, x \in V_k,$$

and reset

$$\mu(U_k \backslash E_k) < \frac{\varepsilon}{3 \cdot 2^{-|k|+k+1}},$$

which leads to our desired conclusion as well.

(3)

Proof. Utilize (1) and (2), we obtain

$$\nu(E) = \inf\{\nu(U) \mid E \subset U, U \text{ open}\} = \inf\{\nu'(U) \mid E \subset U, U \text{ open}\} = \nu'(E)$$

for every Borel set E. □

2.2 Exercise 8.5.35

(1)

Proof. By definition,

$$\varphi_m(f) = S_m f(0) = \int_{\mathbb{T}} f(y) D_m(-y)\,\mathrm{d}y = \int_{\mathbb{T}} f(y) D_m(y)\,\mathrm{d}y.$$

Therefore,

$$|\varphi(f)| \leq \int_{\mathbb{T}} |f(y) D_m(y)|\,\mathrm{d}y \leq \sup_{x\in\mathbb{T}} |f(x)| \int_{\mathbb{T}} |D_m(y)|\,\mathrm{d}y = \|f\|_{C(\mathbb{T})} \|D_m\|_1.$$

In order to achieve the equality, construct a sequence of function $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{T})$ such that $f_n \to \operatorname{sgn}(D_m)$ pointwise, then dominated convergence theorem implies the $\|D_m\|_1$ is exactly the norm. □

Remark 6. *The equality is achieved when $f = sgn(D_m)$ a discontinuous function, instead of $f = 1$, thus we need to approximate it with continuous functions.*

Here we can also apply the ***Riesz representation theorem*** *for Radon measures and rewrite*

$$\varphi_m(f) = \int_{\mathbb{T}} f\,\mathrm{d}\mu$$

for Radon measure $\mathrm{d}\mu = D_m\,\mathrm{d}m$ where m is the Lebesgue measure. Consequently,

$$\|\varphi_m\| = \|\mu\| = \|D_m\|_1.$$

(2)

Proof. Assume the desired set is not meager in $C(\mathbb{T})$, then there is a nonmeager set in which

$$\sup_m |\varphi_m(f)| < +\infty.$$

The resonance theorem implies

$$\sup_m \|D_m\| = \sup_m \|\varphi_m\| < +\infty,$$

a contradiction. □

(3)

Proof. Let $\{r_n\}_{n=1}^{\infty}$ be the sequence of all rational numbers in $(0,1)$, which is dense in $\mathbb{T}$ while countable. Construct an operator

$$\psi_{mn}(f) = S_m f(r_n).$$

Similarly, we can prove that ψ_{mn} is bounded and

$$\|\psi_{mn}\| = \|D_m\|_1.$$

Let $E_n \subset C(\mathbb{T})$ be the meager set that includes every f such that $S_m f(r_n)$ converges as m tends to infinity. As a result, the set including every continuous function that diverges at all rational numbers

$$\bigcap_{n=1}^{\infty} E_n^c = \left(\bigcup_{n=1}^{\infty} E_n\right)^c$$

is the complement of a first category set, hence nonempty by Baire category theorem. ☐

Remark 7. *In comparison with Fourier analysis, it seems more proper to classify this problem into functional analysis.*

2.3 Exercise 8.5.36

Proof. We first prove Fourier transform

$$\begin{aligned}\mathcal{F} : L^1(\mathbb{T}) &\longrightarrow C_0(\mathbb{Z}) \\ f(x) &\longrightarrow \Big(\cdots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \cdots\Big)\end{aligned}$$

is injective. In fact, we only need to show that

$$\hat{f}(n) = 0,\ \forall\, n \in \mathbb{Z} \Longrightarrow f = 0,\ \textit{a.e.}$$

Let $g \in C(\mathbb{T})$, then for every $\varepsilon > 0$, there exists a trigonometric polynomial P such that

$$\sup_{\mathbb{T}} |g - P| < \frac{\varepsilon}{\|f\|_1}.$$

As a result, we have

$$\left|\int_{\mathbb{T}} fg\,\mathrm{d}x\right| \le \left|\int_{\mathbb{T}} fP\,\mathrm{d}x\right| + \int_{\mathbb{T}} |f(g-P)|\,\mathrm{d}x \le \sup_{\mathbb{T}} |g-P| \int_{\mathbb{T}} |f| < \varepsilon.$$

Since g is an arbitrary continuous function on $\mathbb{T}$, f equals 0 almost everywhere.

Assume $\mathcal{F}$ is also surjective, then it is bijective. Given that $\mathcal{F}$ is a bounded linear operator, by the inverse operator theorem we see that $\mathcal{F}^{-1}$ is still a bounded linear operator. Note that

$$\hat{D_m}(n) = \int_{\mathbb{T}} e^{-2\pi inx} \sum_{k=-m}^{m} e^{2\pi ikx} = \begin{cases} 1, & |n| \leq m, \\ 0, & |n| > m, \end{cases}$$

thus

$$\|\mathcal{F}(D_m)\|_{C_0(\mathbb{Z})} = 1.$$

However, it contradicts to the boundedness of $\mathcal{F}^{-1}$ that

$$\|\mathcal{F}^{-1}\| = \|\mathcal{F}^{-1}\| \|\mathcal{F}(D_m)\|_{C_0(\mathbb{Z})} \geq \|\mathcal{F}^{-1}\mathcal{F}(D_m)\|_1 = \|D_m\|_1 \to +\infty$$

as m tends to infinity. Therefore, $\mathcal{F}$ cannot be surjective. □

Remark 8. *After verifying that $\mathcal{F}$ is injective, there are quite a few approach to prove $\mathcal{F}$ is not surjective, including that induced from* ***inverse operator theorem*** *shown above. We shall present* 3 *more profound proofs as follow.*

Proof. **(Duality)**

Assume $\mathcal{F}$ is surjective, then it is a bijection from $L^1(\mathbb{T})$ to $C_0(\mathbb{Z})$. As is known,

$$\|\{\hat{f}(n)\}_{n\in\mathbb{Z}}\|_{C_0(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |\hat{f}(n)| = \left| \int_{\mathbb{T}} f(x) e^{-2n\pi ix} \, \mathrm{d}x \right| \leq \|f\|_{L^1(\mathbb{T})}.$$

Since both $L^1(\mathbb{T})$ and $C_0(\mathbb{Z})$ are complete, the equivalent norm theorem implies $\mathcal{F}$ is an isomorphism from $L^1(\mathbb{T})$ to $C_0(\mathbb{Z})$, which induces an isomorphism between their dual spaces

$$\mathcal{F}^* : L^\infty(\mathbb{T}) \longrightarrow L^1(\mathbb{Z}).$$

However, $L^1(\mathbb{Z})$ is separable while $L^\infty(\mathbb{T})$ is not, a contradiction! □

Proof. **(Termwise Integration)**

For

$$\hat{f}(n) = \begin{cases} 0, & n \leq 0, \\ \frac{1}{\log n}, & n \geq 1, \end{cases}$$

it is obvious that

$$\left(\cdots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \cdots \right) \in C_0(\mathbb{Z}).$$

Assume $f \in L^1(\mathbb{T})$, then the termwise integration theorem of Fourier coefficients implies

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x = \sum_{n=-\infty}^{+\infty} \hat{f}(n) \int_0^{\frac{1}{2}} e^{2n\pi ix}\,\mathrm{d}x = \frac{1}{2\pi i} \sum_{n=1}^{+\infty} \frac{1}{n \log n} \left((-1)^n - 1\right) = +\infty,$$

a contradiction! □

Proof. **(Antisymmetry)**

Construct

$$\left(\cdots, -\frac{1}{\log 3}, -\frac{1}{\log 2}, 0, 0, 0, \frac{1}{\log 2}, \frac{1}{\log 3} \cdots\right) \in C_0(\mathbb{Z}).$$

It cannot be the Fourier coefficients of an L^1 function, otherwise

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} = +\infty,$$

a contradiction! □

3 Littlewood-Paley Theory and Its Applications

3.1 Fourier Multiplier

As Prof. Xi-Nan Ma puts, every PDE in the world is solved by either constructing auxiliary functions and applying maximum principle, or multiplying test functions and integrating by parts. The function multiplied on both sides of equations is call a **multiplier**. As for Fourier analysis, there is a counterpart.

Definition 4 (Fourier multiplier)**.** *For $m \in L^\infty(\mathbb{R}^n)$, define a linear transform on $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ by*

$$(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi),$$

then m is a Fourier multiplier if

$$\|T_m f\|_p \leq C\|f\|_p, \ \forall f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

A typical Fourier multiplier operator is **Hilbert transform** that we introduced in the third tutorial since

$$(Hf)^\wedge(\xi) = (-i\operatorname{sgn}(\xi))\hat{f}(\xi).$$

Here we need a special decomposition to proceed.

Theorem 20 (Radial partition of unity)**.** *There exist $\psi \in C_c^\infty(\mathbb{R}^n)$ such that*

$$\sum_{j=-\infty}^{+\infty} \psi(2^{-j}x) = 1,\ \forall\, x \neq 0.$$

And for a fixed $x \neq 0$, only finitely many terms in this sum is nonzero.

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be the mollifier such that

$$\eta_{B_1(0)} = 1,\ \eta_{B_2(0)^c} = 0,\ 0 \leq \eta \leq 1.$$

Set $\psi(x) = \eta(x) - \eta(2x)$, then

$$\sum_{j=-N}^{N} \psi(2^{-j}x) = \eta(2^{-N}x) - \eta(2^{N+1}x) \to 1$$

as N tends to infinity. It is easy to check that as most two terms are nonzero for any fixed x. □

To prove Littlewood-Paley theorem, we need a theorem concerned with the properties of Fourier multipliers.

Theorem 21 (Hörmander-Mikhlin)**.** *If $m : \mathbb{R}^n \backslash \{0\} \to \mathbb{C}$ satisfies*

$$|\partial^\alpha m(\xi)| \leq B|\xi|^{-\alpha}, \forall, \xi \neq 0$$

for all multi-indices $|\alpha| \leq n+2$, then

$$\|(m\hat{f})^\vee\| \leq CB\|f\|_p,\ \forall 1 < p < +\infty, f \in \mathcal{S}(\mathbb{R}^n).$$

Proof. For $\psi_j(x) = \psi(2^{-j}x)$, set

$$m_j = \psi_j m$$

and consider

$$K = \sum_{j=-N}^{N} K_j = \sum_{j=-N}^{N} \check{m}_j.$$

We shall prove an estimate

$$|\nabla K(x)| \leq CB|x|^{-n-1}$$

where C depends only on dimension n. Actually, the condition implies

$$|\partial^\alpha m_j(\xi)| \le CB2^{-j|\alpha|}$$

and thus

$$\|\partial^\alpha m_j\|_1 \le CB2^{-j|\alpha|}2^{jn}.$$

Similarly, we have

$$\|\partial^\alpha(\xi_i m_j)\|_1 \le CB2^{-j(|\alpha|-1)}2^{jn}.$$

Therefore, the boundedness of Fourier transform implies

$$\|\xi^\alpha \nabla \check{m}_j\|_\infty \le CB2^{-j(|\alpha|-n-1)}.$$

For $|\alpha| = k$, we thus have

$$\|\nabla \check{m}_j\|_\infty \le CB2^{-j(|\alpha|-n-1)}|x|^{-|\alpha|}.$$

Back to the inequality,

$$\begin{aligned}
|\nabla K(x)| &\le \sum_{j=-N}^{N} |\nabla \check{m}_j(x)| \\
&\le \sum_{j:2^{-j}\ge|x|} |\nabla \check{m}_j(x)| + \sum_{j:2^{-j}<|x|} |\nabla \check{m}_j(x)| \\
&\le CB \sum_{j:2^{-j}\ge|x|} 2^{j(n+1)} + \sum_{j:2^{-j}<|x|} 2^{-j}|x|^{-n-2} \\
&\le CB|x|^{-n-1} + CB|x||x|^{-n-2} \\
&= CB|x|^{-n-1}.
\end{aligned}$$

Here C is independent of N since we sum the geometric series before substituting the power of 2 with $|x|$. Therefore, it is correct even if N tends to infinity.

Similarly, we can verify the boundedness condition

$$|K(x)| \le CB|x|^{-n}$$

and the cancellation condition as is recorded in the lecture notes of the third tutorial. Hence K induces a **Caldéron-Zygmund singular integral Operator** T in convolution type, which is strong type (p,p) for $1 < p < +\infty$, and

$$\|(m\hat{f})^\vee\| = \|Tf\|_p \le CB\|f\|_p.$$

□

3.2 Littlewood-Paley Square Function

For $p=2$, Parseval's identity implies

$$\|f\|_2 = \|(\sum |f_j|^2)^{\frac{1}{2}}\|_2$$

where $f_j = f\chi_{[2^j,2^{j+1}]}$. However, such an conclusion fails for lack of inner product structure.

The radical reason is that characteristic functions are "stiff", they cannot cut up a function "regularly". However, sometimes it is necessary to split "low and high frequency term" in the research of hyperbolic equations, especially water wave equations and dispersive equations. As a result, we need a substitute.

Definition 5 (Littlewood-Paley square function)**.** *Define*

$$P_j f = (\psi_j \hat{f})^\vee = f * \check{\psi}_j,$$

and we call

$$Sf = \left(\sum_{j\in\mathbb{Z}} |P_j f|^2\right)^{\frac{1}{2}}$$

the Littlewood-Paley square function of f.

Before introduce the main theorem, we still need a theorem from probability theory.

Theorem 22 (Kkinchin)**.** *Let $\{r_n\}_{n=1}^N$ be a sequence of independent and identically distributed random variables such that*

$$P(w_n = 1) = P(w_n = -1) = \frac{1}{2},$$

then there exist constants c, C depending only on p such that

$$c\left(\sum_{n=1}^N |a_n|^2\right)^{\frac{p}{2}} \le \mathbb{E}\left(\left|\sum_{n=1}^N a_n w_n\right|^p\right) \le C\left(\sum_{n=1}^N |a_n|^2\right)^{\frac{p}{2}}$$

for complex $\{a_n\}_{n=1}^N$.

Proof. We omit the proof here since it is totally an exercise in probability theory. Interested students please refer to Schlag's textbooks. □

Theorem 23 (Littlewood-Paley)**.** *The norms* $\|f\|_p$ *and* $\|Sf\|_p$ *are equivalent, where*

$$\|Sf\|_p = \left(\sum_{j\in\mathbb{Z}} \|f\|_p \right)^{\frac{1}{2}}.$$

Proof. Let $\{w_n\}$ be a sequence of independent and identically distributed random variables such that

$$P(w_n = 1) = P(w_n = -1) = \frac{1}{2},$$

then

$$m(\xi) = \sum_{j=-N}^{N} w_j \psi_j(\xi)$$

satisfies the condition of Hörmander-Mikhlin theorem since

$$\begin{aligned} |\partial^\alpha m(\xi)| &\le \sum_{j\le|N|} |w_j \partial^\alpha \psi_j(\xi)| \\ &\le \sum_{|j|\le N} 2^{-j|\alpha|} |(\partial^\alpha \psi)(2^{-j}\xi)| \\ &\le C \sum_{|j|\le N} |\xi|^{-|\alpha|} |(\partial^\alpha \psi)(2^{-j}\xi)| \\ &\le C|\xi|^{-|\alpha|} \|\partial^\alpha \psi\|_\infty. \end{aligned}$$

By Mikhlin's theorem, we have

$$\begin{aligned} \int |Sf|^p = \int \lim_{N\to\infty} \left(\sum_{|j|\le N} |P_j f|^2 \right)^{\frac{p}{2}} \\ \le \varliminf_{N\to\infty} \int \left(\sum_{|j|\le N} |P_j f|^2 \right)^{\frac{p}{2}} \\ \le C_p \varliminf_{N\to\infty} \int \mathbb{E} \left(\sum_{|j|\le N} |w_j P_j f|^p \right). \end{aligned}$$

By Fubini's theorem, we further have

$$\begin{aligned}
\varlimsup_{N\to\infty} \int \mathbb{E}\left(\sum_{|j|\le N} |w_j P_j f|^p\right) &= \varlimsup_{N\to\infty} \mathbb{E}\left(\int \sum_{|j|\le N} |w_j P_j f|^p\right) \\
&= \varlimsup_{N\to\infty} \mathbb{E}\left(\int \sum_{|j|\le N} |(w_j \psi_j \hat{f})^\vee|^p\right) \\
&\le \varlimsup_{N\to\infty} \mathbb{E}\left(\int \sum_{|j|\le N} |(\psi_j \hat{f})^\vee|^p\right) \\
&\le \|f\|_p^p.
\end{aligned}$$

The conclusion of Hörmander-Mikhlin theorem is applied in the final step.

We shall prove the other inequality by duality. Set another mollifier $\tilde{\psi}$ that equals to 1 on the support of ψ, then

$$\tilde{\psi}_j \psi_j = \psi_j \Longrightarrow \tilde{P}_j P_j = P_j$$

where $\tilde{P}_j f = (\tilde{\psi}_j \hat{f})^\vee$.

For $1 < p < +\infty$ and $f, g \in \mathcal{S}$, we have

$$\begin{aligned}
|\langle f, g\rangle| = \left|\langle \sum_{j\in\mathbb{Z}} P_j f, g\rangle\right| &= \left|\sum_{j\in\mathbb{Z}} \langle P_j f, g\rangle\right| = \left|\sum_{j\in\mathbb{Z}} \langle \tilde{P}_j P_j f, g\rangle\right| = \left|\sum_{j\in\mathbb{Z}} \langle P_j f, \tilde{P}_j g\rangle\right| \\
&\le \left|\int \left(\sum_{j\in\mathbb{Z}} |P_j f|^2\right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{Z}} |\tilde{P}_j g|^2\right)^{\frac{1}{2}}\right| \le \|Sf\|_p \|\tilde{S}g\|_{p'} \le C_p \|Sf\|_p \|g\|_{p'}.
\end{aligned}$$

By the duality property of L^p norm, we have

$$\|f\|_p \le C_p \|Sf\|_p.$$

□

3.3 Applications in Hyperbolic PDEs

In case some students have not mastered the theories of weak solutions, we define general Sobolev spaces.

Definition 6 (Sobolev space)**.** *For multi-index α, if the **weak derivative** $\partial^\alpha u$ exists and belongs to L^p as long as $|\alpha| \le k$, then u belongs the Banach space $W^{k,p}$*

where

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \|\partial^\alpha u\|_p^p \right)^{\frac{1}{p}}.$$

Particularly, $W^{k,2}$ is a Hilbert space, hence we usually denote

$$H^k = W^{k,2}.$$

Remark 9. *If you are not clear about the concept of weak derivatives, just ignore it, which is not essential.*

The space $W^{k,p}$ is indispensable in the theory of elliptic equations. However, some times it is not adequate for hyperbolic equations. As is introduced in class, fractional derivatives are well-defined by Fourier transform.

Definition 7 (Fractional Sobolev space)**.** *For any $s \in \mathbb{R}$, consider the following norms*

$$\|u\|_{H^s} = \|\langle \xi \rangle^s \hat{u}\|_2,$$
$$\|u\|_{\dot{H}^s} = \||\xi|^s \hat{u}\|_2.$$

The fractional Sobolev spaces H^s and $\dot{H}^s$ are respectively the completion of $\mathcal{S}$ under these two norms.

Remark 10. *The definition of H^s is compatible with some general Sobolev spaces as a result of Plancherel theorem of Sobolev inequality (introduced in the fourth tutorial). In fact,*

$$H^s = H_0^k, \ \forall\, s = k = 1, 2, \cdots,$$

where H_0^k is the closure of C_0^∞ functions in H^k.

In fact, $\dot{H}^s$ is a larger function space than H^s as the former somehow admits a worse singularity at origin.

The space $\dot{H}^s$ established the fundamental structure of dispersive equations. Since Sobolev embedding is of great importance in elliptic theories, we wish to construct something similar.

Theorem 24 (Sobolev inequality for H^s)**.** *For $u \in H^s(\mathbb{R}^n)$, we have*

$$\|u\|_p \le C(s, n, p)\|u\|_{H^s}, \ \forall\, 2 \le p \le +\infty, \ s > \frac{n}{2}.$$

Proof. Cauchy inequality implies

$$\|u\|_\infty \le \|\hat{u}\|_1 \le \|\langle\xi\rangle^{-s}\|_2\|\langle\xi\rangle^s\hat{u}\|_2 = C\|u\|_{H^s}.$$

It is obvious that

$$\|u\|_2 \le \|\hat{u}\|_2 \le \|\langle\xi\rangle^s\hat{u}\|_2,$$

thus Riesz-Thorin interpolation theorem implies the theorem. □

Actually, we have similar conclusions for $\dot{H}^s$, however, the argument above fails. The Littlewood-Paley decomposition is a must here.

Theorem 25 (Sobolev inequality for $\dot{H}^s$)**.** *Given* $0 \le s < \frac{d}{2}$ *and*

$$\frac{1}{2} - \frac{1}{p} = \frac{s}{n},$$

we have

$$\|u\|_p \le C(n,p)\|u\|_{\dot{H}^s}$$

for $u \in \dot{H}^s(\mathbb{R}^n)$*.*

Proof. We only need to prove this inequality for $u \in \mathcal{S}(\mathbb{R}^n)$.

Initially, assume $\hat{u}$ is supported in a dyadic annulus

$$\{\xi \mid 2^j \le |\xi| \le 2^{j+1}\}.$$

Since $\hat{u} = \chi_{[2^j,2^{j+1}]}\hat{u}$, Young's inequality implies

$$\|u\|_p \le C(n)2^{jn(\frac{1}{2}-\frac{1}{p})}\|u\|_2 \le C(n)2^{js}\|\hat{u}\|_2 = C(n)\|u\|_{\dot{H}^s}.$$

For general u, apply Littlewood-Paley theorem.

$$\begin{aligned}\|u\|_p^2 &\le C(n,p)\sum_{j\in\mathbb{Z}}\|P_ju\|_p^2\\ &\le C(n,p)\sum_{j\in\mathbb{Z}}\|P_ju\|_{\dot{H}^s}^2\\ &\le C(n,p)\sum_{j\in\mathbb{Z}}\||\xi|^s\psi_j\hat{u}\|_2^2\\ &\le C(n,p)\||\xi|^s\hat{u}\|_2^2\\ &= C(n,p)\|u\|_2^2.\end{aligned}$$

□

Remark 11. *General Sobolev inequality implies Rellish-Kondrachov theorem. Similarly,* $\dot{H}^s$ *is compacted embedded into* L^p *for proper* s *and* p*.*