Advanced Real Analysis Tutorial 03

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1 Review

1.1 More Properties of L^p Space

As a variety of special Banach spaces, L^p spaces in possess of quantities of properties are widely researched in linear functional analysis.

Theorem 1 (Duality)**.** *The* ***dual space*** *of a Banach space X is composed of all bounded linear functionals on X, denoted as X^*. For $1 \le p < +\infty$, the dual space of L^p is* ***isometric*** *with $L^{p'}$. In fact, the linear mapping*

$$\varphi : L^{p'} \to (L^p)^*,$$
$$g \to T_g$$

is an isometry. Here T_g maps f into $\int fg$.

Particularly, the operator norm of T equals the $L^{p'}$ norm of g. In other words

$$\|g\|_{p'} = \|T_g\| = \sup_{\|f\|_p=1} \left| \int fg \right|.$$

This conclusion is correct for $p = +\infty$ as well if the measure is ***semifinite****.*

Remark 1. *This theorem does not apply to L^∞, since $L^1 \subset (L^\infty)^*$ strictly. Actually $(L^\infty(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$. The* ***Bounded Mean Oscillation*** *space is a collection of all L^1_{loc} functions finite under the BMO norm*

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| \, .$$

Here f_Q is the average of f on Q, which is

$$f_Q = \frac{1}{|Q|} \int_Q f,$$

and the supermum is taken over all cubes Q in $\mathbb{R}^n$.

Interested students please refer to other textbooks such as GTM 250.

Other properties also shows the particularity of L^1 and L^∞, the "two ends" of $[1, +\infty]$.

Theorem 2 (Reflexivity)**.** *For $1 < p < +\infty$, the L^p space is reflexive. In other words, L^p is isometric with the $(L^p)^{**}$,* ***the second dual space****.*

Remark 2. *With reflexivity,* ***Banach-Alaoglu theorem*** *admits a weakly convergent subsequence in every bounded sequence.*

Theorem 3 (Separability)**.** *For $1 \le p < +\infty$, the L^p space is separable. In other words, there is a countable subset dense in L^p.*

Remark 3. *In fact, there is a sequence of simple or smooth L^p functions approximate a given L^p function in L^p norm if $p < +\infty$. However, there does not exist a countable subset dense in L^∞.*

In practice, some functions are so irregular that they are not even in L^p. For example, $f(x) = \frac{1}{|x|}$ does not belong to any L^p space. Meanwhile, some common operators, like **Hardy-Littlewood maximal operator**, fail to be strong type (p, p) for every $1 \le p \le +\infty$. Therefore, we introduce a kind of weaker function spaces.

Definition 1 (Weak L^p space)**.** *Define weak L^p norm*

$$\|f\|_{p,\infty} = \left(\sup_{\lambda > 0} \lambda^p \mu\left(\{|f| > \lambda\}\right) \right)^p,$$

then the weak L^p space includes all functions whose weak L^p norms are finite, denoted as $L^{p,\infty}$.

Remark 4. *For $1 \le p \le +\infty$, we have $L^p \subset L^{p,\infty}$. Particularly, $L^\infty = L^{\infty,\infty}$.*

As we know, the essence of Lebesgue integration is splitting the area not vertically but horizontally. There is an important formula that convert an L^p norm into an integral with respect to the **upper level set**.

Theorem 4 (Layer cake representation)**.** *Let f be a L^p function for $1 \le p < +\infty$, then*

$$\int |f|^p \, \mathrm{d}\mu = \int_0^\infty p\lambda^{p-1} \mu\left(\{|f| > \lambda\}\right) \mathrm{d}\lambda.$$

Remark 5. *This widely applied formula that should be born in mind. One possible proof comes from Fubini theorem.*

1.2 L^p Inequalities

Besides Hölder's inequality and its corollaries, there are still a lot of useful inequalities concerned with L^p spaces. To prove that convergence in L^p implies convergence in measure, we need

Theorem 5 (Chebbyshev's inequality)**.** *For $f \in L^p$ and $\alpha > 0$, we have*

$$\mu\left(\{|f| > \lambda\}\right) \le \left(\frac{\|f\|_p}{\lambda} \right)^p.$$

This inequality often take effects on $L^{p,\infty}$.

Besides the triangular inequality in L^p, there is another inequality named after Minkowski.

Theorem 6 (Minkowski's inequality for integrals)**.** *For $1 \leq p < +\infty$ and non-negative f, we have*

$$\left(\int \left(\int f(x,y)\,\mathrm{d}\nu(y)\right)\mathrm{d}\mu(x)\right)^{\frac{1}{p}} \leq \int \left(\int (f(x,y))^p\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(x).$$

Let $f(\cdot,y) \in L^p(\mu)$ for almost every y and $1 \leq p \leq +\infty$. If $y \to \|f(\cdot,y)\|_p$ belongs to $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for almost every x, and

$$\left\|\int f(\cdot,y)\,\mathrm{d}\nu(y)\right\|_p \leq \int \|f(\cdot,y)\|_p\,\mathrm{d}\nu(y).$$

Functions resembling $\frac{1}{x+y}$ usually fail to possess useful integration properties. However, it is improved through multiplying with a L^p functions before integrating. Literally, we call $K(x,y)$ a (-1)-homogeneous function or kernel if $K(x,y) = \lambda K(\lambda x, \lambda y)$ for positive λ.

Theorem 7. *Let (x,y) be a (-1)-homogeneous function such that*

$$\int_0^{+\infty} |K(x,1)| x^{-\frac{1}{p}}\,\mathrm{d}x = C < +\infty,\ p \in [1,+\infty].$$

For $f \in L^p$ and $g \in L^{p'}$, define two operators T and S by

$$Tf(y) = \int_0^\infty K(x,y)f(x)\,\mathrm{d}x,$$
$$Sg(x) = \int_0^\infty K(x,y)g(y)\,\mathrm{d}y,$$

then T and S are bounded. To specify, we have

$$\|Tf\|_p \leq C\|f\|_p,$$
$$\|Sg\|_{p'} \leq C\|g\|_{p'}.$$

Finally, we introduce an inequality that does not look so "regular".

Theorem 8 (Hardy's inequality)**.** *Let*

$$Tf(y) = \frac{1}{y}\int_0^y f(x)\,\mathrm{d}x,$$
$$Sg(x) = \int_x^{+\infty} \frac{g(y)}{y}\,\mathrm{d}y.$$

Then we have the following inequalities for $1 < p \le +\infty$,

$$\|Tf\|_p \le \frac{p}{p-1}\|f\|_p,$$
$$\|Sg\|_{p'} \le p'\|g\|_{p'}.$$

Remark 6. *There is a discrete Hardy's inequality. For* $p > 1$ *and non-negative* $x_1, \cdots, x_n$*, we have*

$$\sum_{k=1}^{n}\left(\frac{1}{k}\sum_{j=1}^{k} x_j\right) \le \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} x_k^p.$$

1.3 Interpolation Theorems

The interpolation theorems play an indispensable role in the boundedness of operators, especially singular integral operators. The proofs are sophisticated, so just remember the statements and applications.

Theorem 9 (Marcinkiewicz interpolation theorem)**.** *Abstract* $1 \le p_0, p_1, q_0, q_1 \le +\infty$ *such that* $p_0 \le q_0, p_1 \le q_1, q_0 \ne q_1$ *and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1},$$
$$\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If a sublinear operator T *is weak type* (p_0, q_0) *and* (p_1, q_1)*, then* T *is strong type* (p, q).

Theorem 10 (Riesz-Thorin interpolation theorem)**.** *Let* T *be a linear operator and* p_0, p_1, q_0, q_1 *defined in the theorem above. If* T *satisfies*

$$\|Tf\|_{q_0} \le M_0\|f\|_{p_0},$$
$$\|Tf\|_{q_1} \le M_1\|f\|_{p_1},$$

then

$$\|Tf\|_q \le M_0^{1-t}M_1^t\|f\|_p$$

Remark 7. *In the latter theorem, we have stricter requirements for* T *but looser requirements for* p_0, p_1, q_0, q_1.

2 Solutions to Homework

2.1 Exercise 6.1.9

Proof. Fix a $\varepsilon > 0$, we have

$$\begin{aligned}\mu\{|f_n - f| \geq \varepsilon\} &= \frac{1}{\varepsilon^p} \int_{\{|f_n - f|^p \geq \varepsilon^p\}} \varepsilon^p \, \mathrm{d}\mu \\ &= \frac{1}{\varepsilon^p} \int_{\{|f_n - f|^p \geq \varepsilon^p\}} |f_n - f|^p \, \mathrm{d}\mu \\ &\leq \frac{1}{\varepsilon^p} \int_X |f_n - f|^p \, \mathrm{d}\mu \\ &\to 0, \ n \to \infty.\end{aligned}$$

Conversely, assume $\{f_n\}_{n=1}^\infty$ does not converge to f in L^p, then there is a subsequence $\{g_n\}_{n=1}^\infty \subset \{f_n\}_{n=1}^\infty$ such that $\exists\, \varepsilon_0 > 0$,

$$\|g_n - f\|_p \geq \varepsilon, \ \forall\, n.$$

Since $g_n \to f$ in measure, there is a subsequence $\{h_n\}_{h=1}^\infty \subset \{g_n\}_{n=1}^\infty$ that converges to f almost everywhere. Dominant convergence theorem implies $h_n \to f$ in L^p, a contradiction. ☐

Remark 8. *Actually, a series is convergent if and only if every subsequence of it possesses a convergent subsequence. We proved this proposition by contradiction.*

2.2 Exercise 6.1.10

Proof. $\Longrightarrow$:

The triangle inequality implies

$$|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p \to 0, \ n \to \infty.$$

$\Longleftarrow$:

As for the inverse proposition, we shall verify a primary inequality first

$$|a \pm b|^p \leq 2^{p-1} \left(|a|^p + |b|^p\right), \ \forall, p \geq 1.$$

In fact, it is equivalent with

$$\left|\frac{a \pm b}{2}\right|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p,$$

a consequence of the convexity of function $f(x) = |x|^p$ for $p \geq 1$.

Back to the point, construct

$$g_n = 2^{p-1}\left(|f_n|^p + |f|^p\right)$$

that converges to $g = |2f|^p \in L^1$ almost everywhere. Since $|f_n - f|^p \leq g_n$, the dominant convergence theorem implies

$$\lim_{n\to\infty} \int |f_n - f|^p = \int \lim_{n\to\infty} |f_n - f|^p = 0 \Longrightarrow \lim_{n\to\infty} \|f_n\|_p = \|f\|_p.$$

□

Remark 9. *A generalization of this conclusion is called* ***Brezis-Lieb Lemma****,*

$$\|u + v_j\|_p^p = \|u\|_p^p + \|v_j\|_p^p + o(1)$$

for $v_j \to 0$ in L^p. It is a fundamental technique in calculus of variations.

2.3 Exercise 6.1.15

Proof. $\Longrightarrow$:

The completeness of L^p space implies $\{f_n\}_{n=1}^{\infty}$ converges to some $f \in L^p$, and the first two conclusions are immediate due to our previous homework.

As for the third, consider an increasing sequence of sets

$$E_m = \left\{|f_n| \geq \frac{1}{m}\right\}$$

for fixed n. Obviously, $\mu(E_m)$ is finite since $f_n \in L^p$. Additionally, note that

$$\int_E |f_n|^p = \int |f_n|^p < +\infty,$$

where

$$E = \bigcup_{m=1}^{\infty} E_m = \{|f_n| \geq 0\}.$$

Due to the fact that $|f_n \chi_{E_m^c}| \leq f_n \in L^p$, we can apply the dominant convergence theorem,

$$\lim_{m\to\infty} \|f_n\|_{L^p(E_m^c)} = \lim_{m\to\infty} \left(\int_{E_m^c} |f_n|^p\right)^{\frac{1}{p}} = \left(\lim_{m\to\infty} \int |f_n|^p \chi_{E_m^c}\right)^{\frac{1}{p}} = \left(\int_{E^c} |f_n|\right)^{\frac{1}{p}} = 0.$$

As a result, $\forall\, \varepsilon > 0$, $\exists\, m > 0$, such that

$$\|f_n\|^p_{L^p(E^c_m)} < \varepsilon \Longrightarrow \int_{E^c_m} |f_n|^p < \varepsilon,$$

while $\mu(E^c_m) < +\infty$.

$\Longleftarrow$:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of L^p functions in possession of the three properties. For a fixed $\forall\, \varepsilon > 0$, consider the sets

$$A_{mn} = E \cap \left\{ |f_m - f_n| \geq \frac{\varepsilon^{\frac{1}{p}}}{3^{\frac{1}{p}} \mu(E)} \right\}.$$

Here E is of finite measure and

$$\int_{E^c} |f_n|^p < \left(\frac{\varepsilon}{3}\right)^p, \ \forall\, n.$$

Provided with such conditions, we have

$$\int_{E \backslash A_{mn}} |f_m - f_n|^p \leq \int_{E \backslash A_{mn}} \frac{\varepsilon}{3\mu(E)} = \frac{\varepsilon \mu(E \backslash A_{mn})}{3\mu(E)} \leq \frac{\varepsilon}{3}.$$

Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy in measure, we assume $\mu(A_{mn}) < \delta$ is small in measure for sufficiently large m, n. As a result,

$$\int_{A_{mn}} |f_m - f_n|^n \leq 2^{p-1} \int_{A_{mn}} |f_m|^p + 2^{p-1} \int_{A_{mn}} |f_n|^p \leq \frac{\varepsilon}{3}.$$

The last inequality is correct for sufficiently small δ as a consequence of the uniform integrability of $\{f_n\}_{n=1}^{\infty}$.

Combine the inequalities above, we ultimately obtain

$$\|f_m - f_n\|_p \leq \left(\int_{E^c} |f_m - f_n|^p + \int_{E \backslash A_{mn}} |f_m - f_n|^p + \int_{A_{mn}} |f_m - f_n|^p \right) \leq \varepsilon.$$

□

Remark 10. *According to the hint, we only need to focus on the reasons why the* 3 *parts are arbitrarily small. It is natural to think of applying the absolute continuity of integral to the construction of* E*.*

However, it is interesting to consider how to construct a sequence of increasing sets of finite measure that approximates X*. Note that* X *is not necessarily a metric space, it is pointless to define a family of homocentric balls* $\{B_n(0)\}_{n=1}^{\infty}$ *with increasing radiuses. Meanwhile, we have no idea whether* X *is* σ*-finite, thus we cannot just claim there are a sequence of sets of finite increasing measure that finally fills* X*.*

Exercise 6.3.29

Proof. Set a (-1)-homogeneous function $K(x,y) = x^{\beta-1}y^{-\beta}\chi_{(0,+\infty)}(y-x)$, which satisfies

$$\int_0^{+\infty} |K(1,y)|y^{-\frac{1}{p}}\,\mathrm{d}y = \int_1^{+\infty} x^{\beta-1-\frac{1}{p}} = \frac{1}{1-\beta p} < +\infty,\ \beta < \frac{1}{p}.$$

For $f(x) = x^\gamma h(x)$, we have

$$\|Tf\|_p \le \frac{1}{1-\beta p}\|f\|_p,\ Tf(x) = \int_0^{+\infty} K(x,y)f(y)\,\mathrm{d}y.$$

By **Theorem 6.20**, we have

$$\int_0^{+\infty}\left(\int_0^{+\infty} x^{\beta-1}y^{\gamma-\beta}h(y)\chi_{(0,+\infty)}(y-x)\,\mathrm{d}y\right)^p\mathrm{d}x \le \frac{1}{(1-\beta p)^p}\int_0^{+\infty} x^{\gamma p}(h(x))^p\,\mathrm{d}x,$$

which implies

$$\int_0^{+\infty} x^{p(\beta-1)}\left(\int_x^{+\infty} y^{\gamma-\beta}h(y)\,\mathrm{d}y\right)^p\mathrm{d}x \le \frac{1}{(1-\beta p)^p}\int_0^{+\infty} x^{\gamma p}(h(x))^p\,\mathrm{d}x.$$

Let $\beta = \gamma = 1 + \frac{r-1}{p}$, and we obtain one of the inequalities. Swapping x and y in K, we similarly achieve the other inequality. □

2.4 Exercise 6.3.29

Proof. Set a (-1)-homogeneous function $K(x,y) = x^{\beta-1}y^{-\beta}\chi_{(0,+\infty)}(y-x)$, which satisfies

$$\int_0^{+\infty} |K(1,y)|y^{-\frac{1}{p}}\,\mathrm{d}y = \int_1^{+\infty} x^{\beta-1-\frac{1}{p}} = \frac{1}{1-\beta p} < +\infty,\ \beta < \frac{1}{p}.$$

For $f(x) = x^\gamma h(x)$, we have

$$\|Tf\|_p \le \frac{1}{1-\beta p}\|f\|_p,\ Tf(x) = \int_0^{+\infty} K(x,y)f(y)\,\mathrm{d}y.$$

By **Theorem 6.20**, we have

$$\int_0^{+\infty}\left(\int_0^{+\infty} x^{\beta-1}y^{\gamma-\beta}h(y)\chi_{(0,+\infty)}(y-x)\,\mathrm{d}y\right)^p\mathrm{d}x \le \frac{1}{(1-\beta p)^p}\int_0^{+\infty} x^{\gamma p}(h(x))^p\,\mathrm{d}x,$$

which implies

$$\int_0^{+\infty} x^{p(\beta-1)}\left(\int_x^{+\infty} y^{\gamma-\beta}h(y)\,\mathrm{d}y\right)^p\mathrm{d}x \le \frac{1}{(1-\beta p)^p}\int_0^{+\infty} x^{\gamma p}(h(x))^p\,\mathrm{d}x.$$

Let $\beta = \gamma = 1 + \frac{r-1}{p}$, and we obtain one of the inequalities. Swapping x and y in K, we similarly achieve the other inequality. □

Remark 11. *The characteristic function $\chi_{[0,1]}(y-x)$ offers an x to the limit of the inner integral.*

2.5 Exercise 6.4.36

Proof. For $q < p$, we have

$$\begin{aligned}\int |f|^q \,\mathrm{d}\mu &= \int_0^{+\infty} q\lambda^{q-1}\mu\left(\{|f| > \lambda\}\right)\mathrm{d}\lambda \\ &\leq q\int_0^1 \lambda^{q-1}\mu\left(\{f \neq 0\}\right)\mathrm{d}\lambda + q\|f\|_{p,\infty}^p \int_1^{+\infty} \lambda^{q-p-1}\,\mathrm{d}\lambda \\ &= \mu\left(\{f \neq 0\}\right) + \frac{q}{p-q}\|f\|_{p,\infty}^p \\ &< +\infty.\end{aligned}$$

For $q > p$, let $\|f\|_\infty = M < +\infty$, then

$$\begin{aligned}\int |f|^q \,\mathrm{d}\mu &= \int_0^{M} q\lambda^{q-1}\mu\left(\{|f| > \lambda\}\right)\mathrm{d}\lambda \\ &\leq q\|f\|_{p,\infty}^p \int_0^M \lambda^{q-p-1}\,\mathrm{d}\lambda \\ &= \frac{q}{q-p}M^{q-p} \\ &< +\infty.\end{aligned}$$

□

Remark 12. *Layer cake representation is always the most immediate formula that bridges strong and weak L^p spaces.*

3 Boundedness Singular Integral Operator

As learned in section 6.3, there are a quantity of L^p inequalities, some of which are concerned with a kernel function $K(x,y)$. In practice, operators in the form of

$$(Tu)(x) = \int K(x,y)u(y)\,\mathrm{d}y$$

are common, like **Green function** and the convolution with **approximation identity**.

Particularly, if $K(x,y)$ admits a singularity, we call T a **singular integral operator**. Cardéron and Zygmund established a systematic theory in singular integral operators, including its boundedness, which is always related to the solvability and uniqueness issue of partial differential equations.

3.1 Caldéron-Zygmund Decomposition

To introduce this theory, we need a basic knowledge of the famous **Caldéron-Zygmund decomposition**, decomposing a function into a good part and a bad yet "small" part.

Theorem 11 (Caldéron-Zygmund decomposition for L^1 functions)**.** *For $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, there is a decomposition $f = g + b$ where*

$$b = \sum_{Q \in B} f\chi_Q,$$

and B is a countable collection of disjoint cubes. Moreover, we have

1. *$|g| \leq \lambda$ almost everywhere;*
2. *the average integral of f is bounded,*
$$\lambda < \frac{1}{|Q|}\int_Q |f| \leq 2^n \lambda;$$
3. *The union of the cubes is not too large,*
$$\left|\bigcup_{Q \in B} Q\right| < \frac{1}{\lambda}\|f\|_{L^1}.$$

Proof. Consider the collection of all binary cubes

$$B_k = \left\{ \prod_{j=1}^{n} [2^k m_j, 2^k(m_j+1)) \,\middle|\, m_1, \cdots, m_n \in \mathbb{Z} \right\}, \ k \in \mathbb{Z}.$$

For $Q_1 \in B_{k_1}$ and $Q_2 \in B_{k_@}$, it is easy to figure out the fact $Q_1 \cap Q_2 = \varnothing$ or one of them contains the other. Meanwhile, the union of B_k over $k \in \mathbb{Z}$ is countable.

Since $f \in L^1$, there is a sufficiently large $k_0 \in \mathbb{Z}$ such that

$$\frac{1}{|Q|}\int_Q |f| \leq \lambda, \ \forall Q \in B_{k_0}.$$

Actually, every $Q \in B_{k_0}$ is composed of 2^n small cubes whose edges are 2^{k_0-1} in length.

Let Q' be one of those small cubes. If

$$\frac{1}{|Q'|}\int_{Q'} |f| > \lambda,$$

then we add Q' to the set B, and notice that

$$\lambda < \frac{1}{|Q'|}\int_{Q'}|f| \le \frac{1}{|Q'|}\int_{Q}|f| \le \frac{\lambda|Q|}{|Q'|} = 2^n\lambda.$$

Otherwise, continue to divide this small cube into 2^n even smaller cubes. This process ends in countable steps.

Ultimately, set

$$b = \sum_{Q\in B} f\chi_Q,\ \ g = f - b,$$

which satisfy the first two requirements with the help of Lebesgue differentiation theorem. Moreover, for $Q \in B$, we have

$$\int_Q |f| > \lambda|Q| \Longrightarrow \left|\bigcup_{Q\in B} Q\right| \le \frac{1}{\lambda}\|f\|_{L^1}.$$

□

3.2 Hilbert Transform

If $K = K(x-y)$, we name T as a **convolutional singular integral operator**, which usually enjoys better properties. The Hilbert transform is a classic convolutional singular integral operator.

Definition 2 (Hilbert transform)**.** *Let $\varphi \in C_c^\infty(\mathbb{R})$, then*

$$H^\varepsilon\varphi(x) = \int_{|x-y|>\varepsilon} \frac{\varphi(y)}{x-y}\,\mathrm{d}y.$$

We define Hilbert transform with the limit

$$H\varphi = \lim_{\varepsilon\to 0^+} H^\varepsilon\varphi.$$

We must verify that H is well-defined in the beginning. Thanks to the smoothness of φ, we have

$$\begin{aligned}|H\varphi(x)| &= \lim_{\varepsilon\to 0^+}\left|\int_{\varepsilon\le|x-y|\le 1}\frac{\varphi(y)}{x-y}\,\mathrm{d}y\right| + \left|\int_{|x-y|>1}\frac{\varphi(y)}{x-y}\,\mathrm{d}y\right| \\ &\le \lim_{\varepsilon\to 0^+}\int_{\varepsilon\le|x-y|\le 1}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|\mathrm{d}y + \int_{|x-y|>1}\left|\frac{\varphi(y)}{x-y}\right|\mathrm{d}y \\ &\le 2\|\varphi'\|_{L^\infty} + \|\varphi\|_{L^1}.\end{aligned}$$

As an application of interpolation theorems, we can obtain the boundedness of Hilbert transform. Beforehand, we need two lemmas.

Lemma 1 (The Fourier transform of Hilbert operator). *For $f \in L^2$, we have*

$$(Hf)^\wedge(\xi) = -isgn(\xi)\hat{f}(\xi).$$

The proof is elementary but complicated, so we omit it here. Interested students please search the friendly website.

Lemma 2 (Multiplication formula of Fourier transform). *Let $f, g \in L^1(\mathbb{R})$, then*

$$\int \hat{f} g = \int f \hat{g}.$$

Proof. By Fubini's theorem, we have

$$\begin{aligned}\int \hat{f}(\xi) g(\xi)\,\mathrm{d}\xi &= \int \left(\int f(x) e^{-2\pi i x \xi}\,\mathrm{d}x \right) g(\xi)\,\mathrm{d}\xi \\ &= \int \left(\int g(\xi) e^{-2\pi i x \xi}\,\mathrm{d}\xi \right) f(x)\,\mathrm{d}x \\ &= \int f(x) \hat{g}(x)\,\mathrm{d}x\end{aligned}$$

□

Lemma 3 (Multiplication formula of Hilbert transform). *Let $\varphi, \psi \in C_c^\infty(\mathbb{R})$, then*

$$\int (Hf) g = -\int f(Hg).$$

Proof. For $\varphi, \psi \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned}\int (H\varphi)\psi = \int (H\varphi)\hat{\check{\psi}} &= \int (H\varphi)^\wedge \check{\psi} \\ &= -i \int \operatorname{sgn}(\xi) \hat{\varphi}(\xi) \check{\psi}(\xi)\,\mathrm{d}\xi \\ &= i \int \operatorname{sgn}(\zeta) \hat{\varphi}(-\zeta) \check{\psi}(-\zeta)\,\mathrm{d}\zeta \\ &= i \int \operatorname{sgn}(\zeta) \check{\varphi}(\zeta) \hat{\psi}(\zeta)\,\mathrm{d}\zeta \\ &= -\int \check{\varphi} (H\psi)^\wedge = -\int \varphi (H\psi).\end{aligned}$$

□

Now we can present a proof of the boundedness of H.

Theorem 12. *Hilbert transform is weak type* $(1,1)$ *and strong type* (p,p) *for* $1 < p < +\infty$.

Proof. For $\varphi \in C_c^\infty$, the first two lemmas imply

$$\|H\varphi\|_{L^2} = \|(H\varphi)^\wedge\|_{L^2} = \|-i\mathrm{sgn}(\xi)\hat{\varphi}(\xi)\|_{L^2} = \|\hat{\varphi}\|_{L^2} = \|\varphi\|_{L^2}.$$

Therefore, the operator $H : L^2 \to L^2$ is an isometry.

Next we shall prove that H is weak type $(1,1)$. Apply Cardéron-Zygmund decomposition to $f \in L^1$, we obtain a collection of countable binary intervals $B = \{I_j\}$, and

$$g(x) = \begin{cases} f(x), & x \notin \bigcup_j I_j \\ \frac{1}{|I_j|}\int_{I_j} f, & x \in I_j, \end{cases}$$

$$b(x) = \sum_j b_j(x) = \sum_j \left(f(x) - \frac{1}{|I_j|}\int_{I_j} f \right) \chi_{I_j}.$$

which is slightly different from the original definition. Here $|g| \leq 2\lambda$ almost everywhere.

Therefore, we split f into two parts

$$|\{|Hf| > \lambda\}| \leq \left|\left\{|Hg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Hb| > \frac{\lambda}{2}\right\}\right|,$$

and estimate them separately.

For the first part, Chebbyshev's inequality implies

$$\left|\left\{|Hg| > \frac{\lambda}{2}\right\}\right| \leq \frac{4}{\lambda^2}\int |Hg|^2 = \frac{4}{\lambda^2}\int |g|^2 \leq \frac{8}{\lambda}\int |g| = \frac{8}{\lambda}\int |f| < +\infty.$$

For the second part, we need more sophisticated estimates. Denote $2I_j$ be the interval homocentric with I_j such that $|2I_j| = 2|I_j|$ and

$$\Omega = \bigcup_j 2I_j, \ |\Omega| \leq 2\left|\bigcup_j I_j\right|$$

It is obvious that

$$\left|\left\{|Hb| > \frac{\lambda}{2}\right\}\right| \leq |\Omega| + \left|\left\{x \notin \Omega \,\middle|\, |Hb(x)| > \frac{\lambda}{2}\right\}\right| \leq \frac{2}{\lambda}\|f\|_{L^1} + \frac{2}{\lambda}\int_{\Omega^c} |Hb|.$$

Additionally, let c_j be the center of $2I_j$, then

$$\int_{\Omega^c} |Hb| = \int_{\Omega^c} \left| H\left(\sum_j b_j\right) \right| \leq \sum_j \int_{\Omega^c} |Hb_j| \leq \sum_j \int_{(2I_j)^c} |Hb_j|,$$

and

$$\begin{aligned}
\int_{(2I_j)^c} |Hb_j| &= \frac{1}{\pi} \int_{(2I_j)^c} \left| \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{b_j(y)}{x-y} \, \mathrm{d}y \right| \mathrm{d}x \\
&= \frac{1}{\pi} \int_{(2I_j)^c} \left| \int_{I_j} \frac{b_j(y)}{x-y} \, \mathrm{d}y \right| \mathrm{d}x \\
&= \frac{1}{\pi} \int_{(2I_j)^c} \left| \int_{I_j} b_j(y) \left(\frac{1}{x-y} - \frac{1}{x-c_j} \right) \mathrm{d}y \right| \mathrm{d}x \\
&\leq \frac{1}{\pi} \int_{I_j} \left(\int_{(2I_j)^c} \left| b_j(y) \left(\frac{1}{x-y} - \frac{1}{x-c_j} \right) \right| \mathrm{d}x \right) \mathrm{d}y \\
&= \frac{1}{\pi} \int_{I_j} |b_j(y)| \left(\int_{(2I_j)^c} \frac{|y-c_j|}{|x-y||x-c_j|} \, \mathrm{d}x \right) \mathrm{d}y \\
&\leq \frac{1}{\pi} \int_{I_j} |b_j(y)| \left(\int_{(2I_j)^c} \frac{|I_j|}{|x-c_j|^2} \, \mathrm{d}x \right) \mathrm{d}y \\
&= \frac{2}{\pi} \int_{I_j} |b_j|.
\end{aligned}$$

The last inequality is a corollary of the fact

$$|y-c_j| \leq \frac{1}{2}|I_j| \Longrightarrow |x-c_j| \leq |y-c_j| + |x-y| \leq \frac{1}{2}|I_j| + |x-y| \leq 2|x-y|.$$

By the definition of b_j, we have

$$\sum_j \int_{(2I_j)^c} |Hb_j| \leq \frac{2}{\pi} \sum_j \int_{I_j} |b_j| \leq \frac{4}{\pi} \sum_j \int_{I_j} |f| \leq \frac{4}{\pi} \|f\|_{L^1}.$$

In summary, we finally obtain

$$\lambda |\{|Hf| > \lambda\}| \leq \left(10 + \frac{8}{\pi}\right) \|f\|_{L^1}.$$

So far, we have shown that H is strong type $(2,2)$ and strong type $(1,1)$. Therefore, Marcinkiewicz interpolation theorem implies H is strong type (p,p) for $1 < p \leq 2$.

When is comes to the case $2 < p < p'$, consider its conjugate index $p' \in (1, 2)$. Recall that we can define L^p norm through duality

$$\begin{aligned}
\|Hf\|_{L^p} &= \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}) \\ \|\varphi\|_{p'} \le 1}} \left| \int (Hf)\varphi \right| \\
&= \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}) \\ \|\varphi\|_{p'} \le 1}} \left| \int f(H\varphi) \right| \\
&\le \|f\|_{L^p} \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}) \\ \|\varphi\|_{p'} \le 1}} \|H\varphi\|_{L^p} \\
&\le C\|f\|_{L^p} \sup_{\substack{\varphi \in C_c^\infty(\mathbb{R}) \\ \|\varphi\|_{p'} \le 1}} \|\varphi\|_{L^p} \\
&= C\|f\|_{L^p}.
\end{aligned}$$

□

Remark 13. *In fact, f is not strong type $(1, 1)$ or (∞, ∞). Direct computation implies*

$$H\chi_{[0,1]} = \frac{1}{\pi} \log \left| \frac{x}{x-1} \right|,$$

which belongs to neither L^1 nor L^∞.

This theorem implies we can extend the domain of H from $C_c^\infty(\mathbb{R})$ to L^p for $1 < p < +\infty$ as a result of density of $C_c^\infty(\mathbb{R})$.

The proof is so comprehensive that it utilized quite a lot of knowledge instructed in this course, including but not limited to

1. definition of strong and weak L^p spaces;
2. Chebbyshev's inequality;
3. monotonous convergence theorem;
4. Fubini's theorem;
5. Interpolation theorem;
6. Duality of L^p;
7. density of C_c^∞;
8. Fourier transform;
9. $\cdots\cdots$

3.3 A Couple of General Theories

Due to time limit, we only introduce some important conclusions on boundedness about general singular integral operators, including convolutional and nonconvolutional types.

Definition 3 (Caldéron-Zygmund kernel)**.** *A function $K(x)$ defined on $\mathbb{R}^n\backslash\{0\}$ is a Caldéron-Zygmund kernel if it satisfies*

1. *(size condition) $|K(x)| \leq B|x|^{-n}$;*
2. *(smoothness condition)*

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)| \,\mathrm{d}x \leq B;$$

3. *(cancellation condition)*

$$\int_{a<|x|<b} K(x)\,\mathrm{d}x = 0, \ \forall\, 0 < r < s < +\infty.$$

*Here B is an absolute constant. Then we define the **convolutional Caldéron-Zygmund operator***

$$Tf(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} K(x-y)f(y)\,\mathrm{d}y.$$

Some theories of Fourier transform imply that T is weak type $(1,1)$ and strong type (p,p) for $1 < p < +\infty$), but not necessarily strong type $(1,1)$ or (∞,∞).

The prove is a bit more interesting. We can prove T is strong type (p,p) for every $1 \leq p < +\infty$, and thus interpolation implies T is strong type (q,q) for $1 < q < p$. Therefore, given a p_0, we can always pick a larger $p > p_0$ to achieve strong type (p_0,p_0).

For the endpoints, we can actually introduce the **Hardy space** and **BMO space**, and figure out $T : H^1 \to L^1$ and $T : L^\infty \to BMO$ are bounded.

Definition 4 (Standard kernel)**.** *A function $K(x,y)$ defined on*

$$(\mathbb{R}^n \times \mathbb{R}^n)\backslash\{(x,x) \mid x \in \mathbb{R}^n\}$$

is a standard kernel if it satisfies

$$K(x,y) \leq \frac{A}{|x-y|^n},$$
$$|K(x_1,y) - K(x_2,y)| \leq \frac{A|x_1-x_2|}{(|x_1-y| + |x_2-y|^{n+\delta})},$$
$$|K(x,y_1) - K(x,y_2)| \leq \frac{A|y_1-y_2|}{(|x-y_1| + |x-y_2|^{n+\delta})},$$

for some constants $A > 0, \delta > 0$. *Define the operator associated with the kernel* $K(x, y)$ *by*

$$Tf(x) = \int K(x, y) f(y) \, \mathrm{d}y.$$

Such singular integral operators of nonconvolution type enjoy similar boundedness properties as we states previously, but the proof is much more challenging. The boundedness issue is always a popular topic in modern Fourier analysis, and also necessary in plenty of PDE theories.