Advanced Real Analysis Tutorial 07

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Contents

1 Review

1.1 Smooth Functions

Intuitively, the space C^∞ is almost the best function space. Additionally, C_c^∞ seems better for it exhibits perfect properties in both differentiation and integration. However, delicacy implies smallness.

As we all know, the pointwise limit of a smooth functions could look terrible. Even uniform limit maintains nothing but continuity. What is worse, it is impossible to endow a norm structure on C^∞, let alone C_c^∞. Therefore, we have to go back to topology to discuss the convergence issue.

Definition 1 (The topology on C^∞). *Let U be an open set. Define a family of semi-norms by*

$$P_{i,j} f = \sup_{x \in K_j} \{ |\partial^\alpha f(x)| \mid |\alpha| \le i \},$$

where $\{K_j\}$ is a series of compact sets that approximates U from the interior.

Such semi-norms induce something analogous to balls in normed spaces, thus we obtain a family of topological basis in $C^\infty(U)$. Conventionally, we denote this topological space as $\mathcal{E}(U)$.

A sequence of functions $\{\varphi_k\}_{k=1}^\infty \subset C^\infty(U)$ converges to φ in $\mathcal{E}(U)$ if and only if

$$\lim_{k \to \infty} \sup_{x \in K} \{ |\partial^\alpha (\varphi_k(x) - \varphi(x))| \mid |\alpha| \le i \}, \ \forall\, \alpha, \forall \text{ compact } K \subset U.$$

Remark 1. *For example, we can take*

$$K_j = \left\{ x \,\middle|\, \inf_{y \in \partial U} |x - y| \ge \frac{1}{j} \right\} \cap \overline{B_j(0)}.$$

In fact, the convolution of a smooth function φ with a family of mollifier converges to φ itself in $\mathcal{E}$.

The compactness of support is usually damaged by taking limits. To maintain the properties of C_c^∞, more restrictions are necessary.

Definition 2 (The topology on C_c^∞). *Let U and $\{K_j\}$ be the sets defined previously. Since*

$$C_c^\infty(U) = \bigcup_{j=1}^\infty \{ \varphi \in C^\infty(U) \mid \mathit{supp}\ \varphi \subset K_j \},$$

the topology in C_c^∞ is induced from that of $\{\varphi \in C^\infty(U) \mid supp\ \varphi \subset K_j\}$. We denote this topological space as $\mathcal{D}(U)$.

A sequence of functions $\{\varphi_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converges to φ in $\mathcal{D}(U)$ if and only if there exists a compact K such that

$$supp\ \varphi_k \subset K, \forall\, k \geq 1,$$

and

$$\lim_{k\to\infty} \sup_{x\in K}\{|\partial^\alpha(\varphi_k(x) - \varphi(x))| \mid |\alpha| \leq i\},\ \forall\, \alpha, \forall\, compact\ K \subset U.$$

Remark 2. *In fact, convergence in $\mathcal{E}$ implies convergence in $\mathcal{D}$ if the supports of all functions locate in a fixed compact set.*

In these two topology, taking derivatives and multiplying a smooth function are continuous operators.

Remark 3. *Since differential operators are continuous on $\mathcal{E}$ and $\mathcal{D}$, they cannot be normed spaces.*

1.2 Three Types of Generalized Functions

By definition, one easily notice that

$$C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n).$$

The duality properties are rather clear in L^p spaces. It is also necessary to describe the dual space of these three spaces, namely continuous linear functionals on them.

Definition 3 (Generalized function)**.** *The space of **distributions** $\mathcal{D}'$, the space of **tempered distribution** $\mathcal{S}'$, and the space of **compactly supported distribution** $\mathcal{E}'$ are respectively the dual of $\mathcal{D}, \mathcal{S}$, and $\mathcal{E}$.*

In fact, we have

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

These three spaces are quite large. Intuitively, it seems difficult to check whether a functional belongs to them.

Definition 4 (Criteria)**.** *We have the following criteria:*

1. *$u \in \mathcal{D}'$ if and only if for any compact K, there exist $m_K \geq 0$ and $C_K \geq 0$ such that*

$$|u(\varphi)| \leq C_K \sup_{\substack{x\in K \\ |\alpha|\leq m_K}} |\partial^\alpha u(\varphi)|,\ \forall\, \varphi \in C_c^\infty \text{ such that } supp\ \varphi \in K;$$

2. $u \in \mathcal{S}'$ *if and only if there exist* $m \geq 0$ *and* $C \geq 0$ *such that*

$$|u(\varphi)| \leq C \sup_{|\alpha|,|\beta| \leq m_K} |x^\alpha \partial^\beta u(\varphi)|, \ \forall \varphi \in \mathcal{S};$$

3. $u \in \mathcal{D}'$ *if and only if there exists a compact* K *and two constants* $m \geq 0$ *and* $C \geq 0$ *such that*

$$|u(\varphi)| \leq C \sup_{\substack{x \in K \\ |\alpha| \leq m}} |\partial^\alpha u(\varphi)|, \ \forall \varphi \in C^\infty.$$

For example the **Dirac mass** δ is a distribution, which satisfies

$$\delta(\varphi) = \varphi(0) = \int \varphi \, \mathrm{d}\delta_0, \ \forall \varphi \in C^\infty.$$

Theorem 1. *Let* $u \in \mathcal{D}'$ *and* $a \in C_c^\infty$*, then* au *is a distribution defined by*

$$(au)(\varphi) = u(a\varphi), \ \forall \varphi \in C^\infty.$$

Similar properties could be developed to $\mathcal{S}'$ and $\mathcal{E}'$.

1.3 Further Properties

In last chapter, we defined the Fourier transform on $\mathcal{S}$. For $u \in \mathcal{S}'$, we can define its Fourier transform by duality. In fact, duality allows one to extend an operator from a good space to a bad one. On some occasions, we use the premodifier "weak" to distinguish them.

The most essential property of smooth functions is their infinite differentiability. It seems that we can transfer derivatives from distributions to smooth functions. To make the boundary term vanish, we conduct this operation on compactly supported functions.

Definition 5 (Distributional derivatives)**.** *For* $u \in \mathcal{D}'$*, define*

$$\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u\left(\partial^\alpha \varphi\right).$$

It is easy to check that $\partial^\alpha u \in \mathcal{D}'$*.*

Remark 4. *In fact, differential operators are continuous on both* $\mathcal{D}$ *and* $\mathcal{D}'$*.*

Derivatives in the sense of distribution are not that practical in the theory of partial differential equations, for we prefer the solution being functions instead of distributions. We need something more precise.

Definition 6 (Weak derivatives). *For $f \in L^1_{loc}(U)$, we call $\partial^\alpha f = g$ in weak sense if*

$$(-1)^{|\alpha|} \int f \partial^\alpha \varphi = \int g\varphi, \ \forall \varphi \in C_c^\infty(U).$$

With the help of approximate identities, we can prove that weak derivatives are unique.

Weak derivative induces **weak solution**, which is a mark of the beginning of modern partial differential equation theories.

In the criterion of distributions, we require $u(\varphi)$ to be bounded by some finite-order derivatives of φ. Thus, we introduce the concept

Definition 7 (The order of a distribution). *A distribution u is of order N if for any compact K, there exists $C_K \geq 0$ such that*

$$|u(\varphi)| \leq C_K \sup_{\substack{x \in K \\ |\alpha| \leq N}} |\partial^\alpha u(\varphi)|, \ \forall \varphi \in C_c^\infty \text{ such that supp } \varphi \in K.$$

It is of infinite order if such N does not exist.

Remark 5. *Here the order N is independent of K, thus there exist infinite-order distributions. By definition, however, every compactly supported distribution is of finite order.*

As we know, the multiplication of two smooth functions is not necessarily integrable. Since a function φ in C_c^∞ is compactly supported, we do not require $u \in \mathcal{D}'$ to be "compactly supported" to make their product integrable. Conversely, we conjecture that the dual of C^∞, namely $\mathcal{E}'$ should be compactly supported. To make sense, we need to figure out the definition of the support of a distribution.

Definition 8 (Support of distributions). *For $u \in \mathcal{D}'$, we say $u = 0$ in an open set Ω if*

$$u(\varphi) = 0, \ \forall \varphi \in C_c^\infty \text{ such that supp } \varphi \subset \Omega.$$

Then the support of u is defined as

$$\left(\bigcup_{\substack{\text{open } \Omega \subset U \\ u=0 \text{ in } \Omega}} \Omega \right)^c.$$

For example, one can verify that supp $\delta = \{0\}$. More essentially,

Theorem 2 (Structure of $\mathcal{E}'$). *Let $\mathcal{D}'_c$ be the collection of compactly supported distribution, then it is isomorphic to $\mathcal{E}'$.*

Remark 6. *This is the reason why we call $\mathcal{E}'$ compactly supported distributions initially.*

2 Solutions to Homework

2.1 Exercise 9.1.9

(1)

Proof. For $\varphi \in C_c^\infty$,

$$\langle \delta \circ S_r, \varphi \rangle = \int_{\mathbb{R}^n} \delta(rx)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \delta(x)\varphi\left(\frac{x}{r}\right) r^{-n}\,\mathrm{d}x = r^{-n}\varphi(0) = r^{-n}\langle \delta, \varphi\rangle.$$

□

(2)

Proof. For $\varphi \in C_c^\infty$,

$$\begin{aligned}
\langle (\partial^\alpha F) \circ S_r, \varphi \rangle &= \int_{\mathbb{R}^n} (\partial^\alpha F)(rx)\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha F(x)\varphi\left(\frac{x}{r}\right) r^{-n}\,\mathrm{d}x \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^n} F(x)\partial^\alpha\left(\varphi\left(\frac{x}{r}\right)\right) r^{-n}\,\mathrm{d}x \\
&= (-1)^{|\alpha|}\int_{\mathbb{R}^n} F(x)(\partial^\alpha\varphi)\left(\frac{x}{r}\right) r^{-n-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha F(x)\varphi\left(\frac{x}{r}\right) r^{-n-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \partial^\alpha(F(rx))\varphi\left(\frac{x}{r}\right) r^{-|\alpha|}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} (\partial^\alpha F)(rx)\varphi\left(\frac{x}{r}\right) r^{\lambda-|\alpha|}\,\mathrm{d}x \\
&= r^{\lambda-|\alpha|}\langle \partial^\alpha F, \varphi\rangle.
\end{aligned}$$

□

(3)

Proof. For $\varphi \in C_c^\infty$,

$$\begin{aligned}
\langle (\chi_{(0,+\infty)} \log x)' \circ S_r, \varphi \rangle &= \int \left(\chi_{(0,+\infty)} \log x\right)' (rx)\varphi(x)\,\mathrm{d}x \\
&= -\int \chi_{(0,+\infty)} \log(rx)\varphi'(x)\,\mathrm{d}x \\
&= -\int \chi_{(0,+\infty)}(\log r + \log x)\varphi'(x)\,\mathrm{d}x \\
&= -\int \chi_{(0,+\infty)}(\log r)\varphi'(x)\,\mathrm{d}x - \int \chi_{(0,+\infty)}(\log x)\varphi'(x)\,\mathrm{d}x \\
&= \int \left(\chi_{(0,+\infty)} \log x\right)' \varphi(x)\,\mathrm{d}x - \int_0^{+\infty} (\log r)\varphi'(x)\,\mathrm{d}x \\
&= \langle (\chi_{(0,+\infty)} \log x)', \varphi \rangle + (\log r)\varphi(0).
\end{aligned}$$

Therefore, it is a nonhomogeneous distribution.

It is easy to check x^{-1} is a homogeneous distribution of degree -1 on $(0, +\infty)$. For $\psi \in C_c^\infty(0, +\infty)$, we have

$$\begin{aligned}
\langle (\chi_{(0,+\infty)} \log x)' - x^{-1}, \psi \rangle &= \int_{(0,+\infty)} (\chi_{(0,+\infty)} \log x)'\psi(x)\,\mathrm{d}x - \int_{(0,+\infty)} x^{-1}\psi(x)\,\mathrm{d}x \\
&= \int_{(0,+\infty)} (\chi_{(0,+\infty)} \log x)'\psi(x)\,\mathrm{d}x - \int_{(0,+\infty)} x^{-1}\psi(x)\,\mathrm{d}x \\
&= \int_{(0,+\infty)} (\log x)\psi'(x)\,\mathrm{d}x - \int_{(0,+\infty)} \frac{\psi(x)}{x}\,\mathrm{d}x \\
&= -\int_{(0,+\infty)} \frac{\psi(x)}{x}\,\mathrm{d}x - \int_{(0,+\infty)} \frac{\psi(x)}{x}\,\mathrm{d}x \\
&= 0.
\end{aligned}$$

□

Remark 7. *A radical reason for this phenomenon is that*

$$C_c^\infty(\mathbb{R}\backslash\{0\}) \subset C_c^\infty(\mathbb{R}).$$

More precisely, compactly supported smooth functions on $\mathbb{R}\backslash\{0\}$ *must vanish in a open neighborhood of* 0*, which eliminates the potential singularity*

2.2 Exercise 9.1.14

(1)

Proof. By direct calculation,

$$F_i^\varepsilon(x) = \frac{1}{\omega_n}\left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}} x_i,$$
$$F_{ii}^\varepsilon(x) = \frac{1}{\omega_n}\left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}} - \frac{n}{\omega_n}\left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}-1} x_i^2,$$

which imply

$$\Delta F^\varepsilon(x) = \sum_{i=1}^n F_{ii}^\varepsilon(x) = \frac{n\varepsilon^2}{\omega_n}\left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}-1} = \frac{1}{\varepsilon^n} g\left(\frac{x}{\varepsilon}\right).$$

□

(2)

Proof. By direct calculation,

$$\int g = \frac{n}{\omega_n}\int \left(|x|^2+1\right)^{-\frac{n+2}{2}} \mathrm{d}x = n\int_0^{+\infty} \frac{r^{n-1}}{(r^2+1)^{\frac{n+2}{2}}} \mathrm{d}r. \quad (1)$$

Let $s = \frac{r^2}{r^2+1}$, then

$$r^2 = \frac{s}{1-s},\ 2r\,\mathrm{d}r = \frac{1}{(1-s)^2}\mathrm{d}s.$$

Change the variable, and

$$\int g = \frac{n}{2}\int_0^{+\infty} \frac{r^{n-2}}{(r^2+1)^{\frac{n+2}{2}}} \mathrm{d}r^2 = \frac{n}{2}\int_0^1 s^{\frac{n-2}{2}} \mathrm{d}s = 1.$$

□

(3)

Proof. We only need to prove $F^\varepsilon \to F$ in D' as $\varepsilon \to 0$. In fact, $F^\varepsilon \to F$ almost everywhere and

$$|F^\varepsilon| \le F \in L^1_{loc}.$$

By previous exercises, F^ε converges to F in distribution. □

(4)

Proof. The condition $\varphi \in C_c^\infty$ implies

$$F * \partial^\alpha \varphi \in L^1$$

for any multi-index α. As a result,

$$\begin{aligned}
\Delta f(x) &= \Delta_x \int F(y)\varphi(x-y)\,\mathrm{d}y \\
&= \int F(y)\Delta_x\varphi(x-y)\,\mathrm{d}y \\
&= \int F(y)\Delta_y\varphi(x-y)\,\mathrm{d}y \\
&= \int \Delta_y F(y)\varphi(x-y)\,\mathrm{d}y \\
&= \int \delta(y)\varphi(x-y)\,\mathrm{d}y \\
&= \varphi(x).
\end{aligned}$$

□

(5)

Proof. we only need to prove (3) for $n = 1, 2$.

If $n = 1$, then $F(x) = \frac{1}{2}|x|$, and

$$\begin{aligned}
\langle \Delta F, \varphi \rangle &= \frac{1}{2}\int |x|\Delta\varphi(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{+\infty} x\varphi''(x)\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^0 x\varphi''(x)\,\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{+\infty} \varphi'(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^0 \varphi'(x)\,\mathrm{d}x \\
&= -\frac{1}{2}\int_0^{+\infty} \varphi'(x)\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^0 \varphi'(x)\,\mathrm{d}x \\
&= \varphi(0)
\end{aligned}$$

for $\varphi \in C_c^\infty(\mathbb{R})$. Therefore, we obtain $\Delta F = \delta$.

If $n = 2$, we can similarly prove (1) and (2), thus (3) is still correct. □

Remark 8. *This is actually the construction of* ***Green's Function*** *on a ball.*

2.3 Exercise 9.2.27

(1)

Proof. Abstract

$$h_\alpha(x) = \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}|x|^{-\alpha},$$

which belongs to $L^1 + L^2$ when $\frac{n}{2} < \alpha < n$. Therefore, $\hat{h}_\alpha$ is well defined. By the definition of Gamma function, we have

$$\begin{aligned}\int h_\alpha(x)e^{-\pi|x|^2}\,\mathrm{d}x &= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{\alpha}{2}}}\int_0^{+\infty} r^{n-\alpha-1}e^{-\pi r^2}\,\mathrm{d}r\\ &= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\int_0^{+\infty} t^{\frac{n-\alpha-2}{2}}e^{-t}\,\mathrm{d}t\\ &= \frac{|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\\ &= \int h_{n-\alpha}(x)e^{-\pi|x|^2}\,\mathrm{d}x.\end{aligned}$$

For

$$\alpha \in \{z \in \mathbb{C} \mid 0 < \mathrm{Re} < n\},$$

it is easy to verify

$$\frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}\int |x|^{-\alpha}e^{-\pi|x|^2}\,\mathrm{d}x$$

and

$$\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n-\alpha}{2}}}\int |x|^{\alpha-n}e^{-\pi|x|^2}\,\mathrm{d}x$$

are holomorphic functions with respect to α that coincide on $[\frac{n}{2}, n] \subset \mathbb{R}$. By the uniqueness theorem, they coincide everywhere in the domain of definition. □

(2)

Proof. Let φ be an arbitrary Schwarz function, then

$$\begin{aligned}|\langle R_\alpha, \varphi\rangle| &\le \left|\frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\right|\int |x|^{a-n}|\varphi(x)|\,\mathrm{d}x\\ &\le C\|\varphi\|_\infty\int_{[-1,1]}|x|^{a-n} + C\int_{[-1,1]^c}\frac{1}{|x|^{2n-a}}|x|^n|\varphi(x)|\,\mathrm{d}x\\ &\le C\|\varphi\|_\infty + C\|x^n\varphi\|_\infty.\end{aligned}$$

Therefore, R_α is a tempered distribution. Note that (1) implies

$$\langle \hat{R}_\alpha, \varphi \rangle = \langle R_\alpha, \hat{\varphi} \rangle = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \langle |x|^{\alpha - n}, \hat{\varphi} \rangle = \frac{1}{(2\pi)^\alpha} \langle |\xi|^{-\alpha}, \varphi \rangle,$$

hence

$$\hat{R}_\alpha = (2\pi|\xi|)^{-\alpha}.$$

□

(3)

Proof. Let φ be an arbitrary Schwarz function, then

$$\int -\Delta R_2 \varphi \, \mathrm{d}x = \int (-\Delta R_2)^\wedge \check{\varphi} \, \mathrm{d}\xi = \int 4\pi^2 |\xi|^2 \hat{R}_2 \check{\varphi} \, \mathrm{d}\xi = \int \check{\varphi} \, \mathrm{d}\xi = (\check{\varphi})^\wedge(0) = \varphi(0).$$

which implies

$$\Delta R_2 = -\delta.$$

□

Remark 9. *This exercise implies some more profound result. By Fourier transform and duality, we can define the **fractional Laplacian** by*

$$(-\Delta)^\alpha f(x) = \left((4\pi^2 |\xi|^2)^\alpha \hat{f}(\xi) \right)^\vee (x) = (4\pi^2)^\alpha (|\xi|^{2\alpha})^\wedge * f$$

UNFINISHED

2.4 Exercise 9.3.36

Proof. Note that

$$\begin{aligned}
\|\varphi_j\|_{H^s} &= \left(\int (1 + |\xi|^2)^s \left| \int \varphi(x - a_j) e^{-2\pi i x \xi} \, \mathrm{d}x \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \left(\int (1 + |\xi|^2)^s \left| \int \varphi(x) e^{-2\pi i (x + a_j)\xi} \, \mathrm{d}x \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \left(\int (1 + |\xi|^2)^s \left| e^{2\pi i a_j \xi} \hat{\varphi}(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \left(\int (1 + |\xi|^2)^s \left| \varphi(\xi) \right|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} \\
&= \|\varphi\|_{H^s} < +\infty.
\end{aligned}$$

This bound is independent of j.

Assume there is a convergent subsequence. Without loss of generality suppose $\{\varphi_j\}_{j=1}^{\infty}$ itself converges to φ in H^s. Since φ is compactly supported, there is an $N > 0$ such that

$$\text{supp } \varphi_j \cap \text{supp } \varphi = (a_j + \text{supp } \varphi) \cap \text{supp } \varphi = \varnothing$$

for any $j \geq N$. In that case,

$$\begin{aligned}
\|\varphi_j - \varphi\|_{H^s}^2 &= \int (1+|\xi|^2)^s \left|(\varphi_j - \hat{\varphi})^\wedge(\xi)\right|^2 \mathrm{d}\xi \\
&= \int (1+|\xi|^2)^s \left|\hat{\varphi}_j(\xi) - \hat{\varphi}(\xi)\right|^2 \mathrm{d}\xi \\
&= \int (1+|\xi|^2)^s |\hat{\varphi}_j(\xi)|^2 \mathrm{d}\xi + \int (1+|\xi|^2)^s |\hat{\varphi}(\xi)|^2 \mathrm{d}\xi \\
&= 2\|\varphi\|_{H^s}^2,
\end{aligned}$$

thus

$$0 = \lim_{j\to\infty} \|\varphi_j - \varphi\|_{H^s} = \|\varphi\|_{H^s} \Longrightarrow \varphi = 0,$$

a contradiction! □

Remark 10. *The essence of this solution is to separate the supports of φ and φ_j.*

Another approach is to show its convergence of φ_j in $L^2(B_1(0))$ or $L^2(B_1^c(0))$ (depending on the sign of s), and thus there exists a subsequence that converges to φ almost everywhere. By Riemann-Lebesgue lemma, it converges to 0 pointwise, which is impossible.

3 Hardy Space and BMO Space

The theory of Hardy and BMO space is complicated and sophisticated. We aim to present their fine properties thus might omit some boring proofs.

3.1 Hardy space

The theory of Hardy spaces is established by Hardy when researching complex analysis in the early 1900s. During the development of theories in function spaces, the manners mathematicians applied to describe Hardy space differ.

In this course, we have learned that some important operators such as Hardy-Littlewood maximal operator, are of weak type $(1,1)$. In other words, they might map an L^1 function into a function excluded from L^1.

To make it easier, a natural idea is to construct a "smaller" space. First, we introduce an essential concept.

Definition 9 (Poisson kernel)**.** *The Poisson kernel is the function*

$$P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}.$$

One can check that $P_\varepsilon(x) = \frac{1}{\varepsilon^n} P(\frac{x}{\varepsilon})$ *is a family of approximate identity.*

Given a tempered distribution u, the convolution $P_t * u$ is a distribution defined by

$$\langle P_t * u, \varphi \rangle = \langle \tilde{\varphi} * u, \tilde{P}_t \rangle$$

where

$$\tilde{\varphi}(x) = \varphi(-x), \; \tilde{P}_t(x) = P_t(-x),$$

since $P_t \in L^1$.

Remark 11. *The convolution of a generalized function and an appropriate function is still a generalized function, which is similarly defined. The identity is always correct when substituting the generalized function with a function in common sense.*

Next, we present the concrete definition of Hardy space.

Definition 10 (Hardy space)**.** *Define the* ***Poisson maximal operator*** *by*

$$M_P f(x) = \sup_{\varepsilon > 0} |P_\varepsilon * f(x)|, \; f \in \mathcal{S}'.$$

We say that $f \in H^p$ *if*

$$\|f\|_{H^p} = \|M_P f\|_{L^p} < +\infty.$$

In fact, H^p spaces are not essential when $1 < p < +\infty$.

Theorem 3 (Equivalence)**.** *For an appropriate function* f *and* $p \in (1, +\infty)$*, we have*

$$\|f\|_{L^p} \le \|f\|_{H^p} \le C_{n,p} \|f\|_{L^p}.$$

Proof. Since $\{P_\varepsilon\}_{\varepsilon>0}$ is a family of approximate identity, we have

$$\lim_{\varepsilon\to 0}\|P_\varepsilon * f - f\|_{L^p} = 0,$$

which implies

$$\|f\|_{L^p} \leq \sup_{\varepsilon>0}\|P_t * f\|_{L^p} = \|\sup_{\varepsilon>0}|P_t * f|\|_{L^p} = \|f\|_{H^p}.$$

The converse inequality is a result of the L^p boundedness of the Hardy-Littlewood maximal operator,

$$\|f\|_{H^p} = \|\sup_{\varepsilon>0}|P_t * f|\|_{L^p} \leq \|Mf\|_{L^p} \leq C_{n,p}\|f\|_{L^p}.$$

□

Remark 12. *For $p = 1$, this argument leads to*

$$\|f\|_{L^1} \leq \|f\|_{H^1},$$

which implies H^1 is a subspace of L^1.

According to this theorem, H^p and L^p are identical when $1 < p < +\infty$. Therefore, we usually focus on the case $0 < p \leq 1$.

There is an equivalent description of Hardy space

Theorem 4 (Littlewood-Paley)**.** *For $p \in (0, +\infty)$, the H^p norm is equivalent with*

$$\left\|\left(\sum_{j\in\mathbb{Z}}|P_j f|^2\right)^{\frac{1}{2}}\right\|_{L^p}.$$

This accounts for the equivalence of H^p and L^p for $1 < p < +\infty$ as well.

Particularly, there is a special property of H^1. It is smaller than L^1 and

Theorem 5 (Boundedness)**.** *The Hardy-Littlewood maximal operator and Hilbert transform are (H^1, L^1) bounded, namely*

$$\|Tf\|_{L^1} \leq C\|f\|_{H^1},\ \forall f \in H^1.$$

The proof is complicated, thus we omit it here.

Theorem 6 (H^1 space)**.** *Let H be the Hilbert transform, then*

$$\|f\|_{H^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}.$$

Remark 13. *We can extend this theorem to higher dimensions by substituting Hilbert transform with* ***Riesz transform****, which is an extension of Hilbert transform to higher dimensions.*

As an ending of this part, we show that H^1 is exactly smaller than L^1.

Theorem 7. *H^1 is a proper subspace of L^1.*

Proof. Given a function $f \in H^1$, previous theorems imply $Hf \in L^1$, thus $(Hf)^\wedge$ is uniformly continuous. Since

$$(Hf)^\wedge(\xi) = -i\pi \mathrm{sgn}(\xi)\hat{f}(\xi),$$

the continuity at origin requires that $\hat{f}(0) = 0$, namely

$$\int f = 0.$$

As a result, $L^1(\mathbb{R}) \backslash H^1(\mathbb{R}) \neq \varnothing$. The same conclusion is correct for higher dimensions. □

3.2 Bounded Mean Oscillation Space

We never worry about Hardy-Littlewood maximal operator in the other endpoint since it is strong type (∞, ∞). However, other operators, such singular integrals, are not that lucky. For example, it is obvious that $\chi_{[0,1]} \in L^\infty$, but one can verify that $Hf \notin L^\infty$.

As a result, we need a "larger" space than L^∞.

Definition 11 (BMO space)**.** *Let Q be a cube in $\mathbb{R}^n$. The average of a function f on Q is defined as*

$$f_Q = \frac{1}{|Q|} \int_Q f.$$

Then we introduce the ***mean oscillation***

$$\frac{1}{|Q|} \int_Q |f - f_Q|.$$

A function f is of bounded mean oscillation if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < +\infty.$$

Note that $\|f\|_{BMO} = 0$ if and only if f is constant almost everywhere. Therefore, we always identify two functions if their difference is constant almost everywhere. In this case, $\|\cdot\|_{BMO}$ is a norm, and the space $(BMO, \|\cdot\|_{BMO})$ is actually a Banach space.

Remark 14. *Of course, we can define BMO by balls rather than cubes.*

Next, we shall show that BMO is definitely larger.

Theorem 8. *L^∞ is a proper subset of BMO.*

Proof. For $f \in L^\infty$, we have

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q| \leq \sup_Q \frac{1}{|Q|}\int_Q |f| + \sup_Q \frac{1}{|Q|}\int |f_Q| \leq 2\|f\|_{L^\infty}.$$

It is obvious that $\log|x| \notin L^\infty$. Next, we shall prove $\log|x| \in BMO$. Define a constant

$$C_{x_0,R} = \begin{cases} \log|x_0|, & |x_0| > 2R, \\ \log R, & |x_0| \leq 2R. \end{cases}$$

By direct computations, $|x_0| > 2R$ implies

$$\begin{aligned} \frac{1}{B_R(x_0)}\int_{B_R(x_0)} |f - C_{x_0,R}| &= \frac{1}{|\mathbb{S}^{n-1}|R^n}\int_{B_R(x_0)} \left|\log\frac{|x|}{|x_0|}\right| \\ &\leq \max\left\{\log\frac{3}{2}, -\log\frac{1}{2}\right\} \\ &< +\infty; \end{aligned}$$

while $|x_0| \leq 2R$ implies

$$\begin{aligned} \frac{1}{B_R(x_0)}\int_{B_R(x_0)} |f - C_{x_0,R}| &= \frac{1}{|\mathbb{S}^{n-1}|R^n}\int_{B_R(x_0)} \left|\log\frac{|x|}{R}\right| \\ &\leq \frac{1}{|\mathbb{S}^{n-1}|R^n}\int_{B_{3R}(0)} \left|\log\frac{|x|}{R}\right| \\ &\leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{B_3(0)} \left|\log\frac{|}{x}|\right| \\ &< +\infty. \end{aligned}$$

In summary,

$$\begin{aligned} \|\log|x|\|_{BMO} &= \sup_B \frac{1}{|B|}\int_B |\log|x| - (\log|x|)_B| \\ &\leq \sup_B \frac{1}{|B|}\int_B |\log|x| - C_{x_0,R}| + \sup_B \frac{1}{|B|}\int_B |(\log|x|)_B - C_{x_0,R}| \\ &< +\infty. \end{aligned}$$

□

Recall **Calderón-Zygmund singular integral operators in convolution type**, which is strong type (p, p) for $1 < p < +\infty$.

Theorem 9 (BMO boundedness)**.** *The Hilbert transform in* (L^∞, BMO) *bounded, namely*

$$\|Hf\|_{BMO} \leq C\|f\|_{L^\infty}.$$

After the following definition, one figures that BMO and L^∞ are not entirely different.

Definition 12 (Sharp function)**.** *For* $f \in BMO$*, define*

$$f^\sharp(x) = \sup_{Q \ni x} \frac{1}{Q} \int_Q |f(y) - f_Q| \, \mathrm{d}y.$$

It satisfies

$$\|f\|_{BMO} = \|f\|_{L^\infty}.$$

With its assistance, we can extend some theories from L^∞ to BMO.

Theorem 10 (Interpolation inequality)**.** *For* $1 \leq< p < q < +\infty$*, a function* $f \in L^p \cap BMO$ *satisfies*

$$\|f\|_{L^q} \leq C\|f\|_{L^p}^{\frac{p}{q}} \|f\|_{BMO}^{\frac{q-p}{q}}.$$

Theorem 11 (Interpolation of operators)**.** *For* $1 \leq p_0 < +\infty$*, if linear operator* T *is strong type* (p_0, p_0) *and* (L^∞, BMO) *bounded, then*

$$\|T\|_{p \to p} \leq C_{n,p,p_0}, \ \forall \, p_0 < p < +\infty.$$

3.3 Duality

Theorem 12 ($H^1 - BMO$ duality)**.** *Let* H^1 *be the Hardy space, then*

$$(H^1)^* = BMO.$$

Remark 15. *Similar to the fact that* $(L^1)^* = L^\infty, (L^\infty)^* \neq L^1$*, the inverse proposition is incorrect.*