Advanced Real Analysis Tutorial 01

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1 Review

1.1 Set Theory Revisited

Operations of sets are trivial, here we only state one theorem that is frequently utilized in this chapter.

Theorem 1 (De Morgan). *Let $\{E_i\}_{i\in I}$ be a family of sets (not necessarily countable), we have the following formulas,*

$$\left(\bigcup_{i\in I} E_i\right)^c = \bigcap_{i\in I} E_i^c, \qquad \left(\bigcap_{i\in I} E_i\right)^c = \bigcup_{i\in I} E_i^c.$$

A more important concept is **the limit of a sequence of sets**, which is the limit of characteristic functions in essence.

Definition 1 (Characteristic function). *The characteristic function of a set E is*

$$\chi_E = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Given a sequence of **increasing sets** $\{E_n\}_{n=1}^{\infty}$ such that $E_1 \subset E_2 \subset \cdots$, their characteristic functions $\{\chi_{E_n}\}_{n=1}^{\infty}$ are increasing with respect to the lower index n while bounded. According what we learned in Mathematical Analysis, the limit function exists and is still a characteristic function χ_E. Correspondingly, the there is a limit set E, written as

$$\lim_{n\to\infty} E_n = E.$$

Similarly, we can define the limit of **decreasing sets**.

What if $\{\chi_{E_n}\}_{n=1}^{\infty}$ has no limit? We can define its **upper and lower limit** in the same manners. For functions, we have

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, \chi_{E_n} &= \{x \mid \forall\, N > 0, \exists\, n > N, \text{such that } \chi_{E_n}(x) = 1\}, \\ \underline{\lim_{n\to\infty}}\, \chi_{E_n} &= \{x \mid \exists\, N > 0, \forall\, n > N, \text{such that } \chi_{E_n}(x) = 1\}. \end{aligned}$$

When it comes to sets,

Definition 2 (Upper and lower limit of sets). *For a sequence of sets $\{\chi_{E_n}\}_{n=1}^{\infty}$,*

$$\begin{aligned} \overline{\lim_{n\to\infty}}\, E_n &= \lim_{n\to\infty} \sup_{k\geq n} E_k = \lim_{n\to\infty} \bigcup_{k=n}^{\infty} E_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k, \\ \underline{\lim_{n\to\infty}}\, E_n &= \lim_{n\to\infty} \inf_{k\geq n} E_k = \lim_{n\to\infty} \bigcap_{k=n}^{\infty} E_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k. \end{aligned}$$

To have a better comprehension, we can regard "$\cup$" as "$\exists$" while "$\cap$" as "$\forall$". Taking lower limit as an example, the latter intersection means $x \in E_k, \forall\, k \geq n$ some sufficiently large n. The former union means we only need the existence of such an n, regardless of how large it is.

1.2 Algebra and σ-Algebra

Measure and integration are the twin tower of real analysis. As is learned in undergraduate Real Analysis, measure theory is establish on a σ-algebra instead of the universal set. Thus, we need to understand some relevant concepts.

Definition 3 (Algebra)**.** *A collection of set $\mathcal{C}$ is an algebra if it is closed under **finite union** and **complement**.*

Simple set operation implies an algebra is also closed under **finite intersection** and **difference**. However, what we need in measure theory is countable operation. Strengthening "finite" to "countable", we obtain the **σ-algebra**.

We always expect union, intersection and complement, but sometimes the collection does not have such satisfactory properties. With two out of the three property, we can deduce the rest one. So weaker structures possess as most one of the three properties. A common structure is **semi-algebra**, where E^c equals a finite union of elements in $\mathcal{C}$. Semi-algebra is widely used in ergodic theory.

Another structure is **ring**.

Definition 4 (Ring)**.** *A collection of set $\mathcal{C}$ is a ring if it is closed under **finite intersection** and **difference**.*

In comparison with algebra, it has somehow a weaker property. Similarly, we can define **σ-ring** and **semi-ring**. Ring is far less useful than algebra in real analysis. Additionally, we could degenerate a ring to a **π-system** by only maintain the intersection property.

In the field of analysis, we always focus on "limit". Obviously for a σ-algebra, we can take limit on a monotonous sequence. But sometimes, we wish this property maintains in a weaker structure.

Definition 5 (Monotone class)**.** *A collection of sets $\mathcal{C}$ is a monotone class if it is closed under the limit of monotonous sequences.*

A direct corollary is that a σ-algebra is a monotone class, and **λ-system** is something lies between the former two structures (a decreasing sequence is a increasing one combined with complement).

Remark 1. *Rings, π-systems, λ-systems are more useful in probability theory.*

Next theorem is significant.

Theorem 2 (Monotone class theorem)**.** *Let $\mathcal{C}$ be a collection of sets.*

1. *$m(\mathcal{C}) = \sigma(\mathcal{C})$ if $\mathcal{C}$ is an algebra;*
2. *$m(\mathcal{C}) = \lambda(\mathcal{C})$ if $\mathcal{C}$ is a π-system.*

Here $m(\mathcal{C}), \sigma(\mathcal{C}), \lambda(\mathcal{C})$ are respectively the monotone class, σ-algebra, λ-system generated by $\mathcal{C}$ (the smallest structure that contains $\mathcal{C}$).

As a vital application, when checking some property of a σ-algebra $\mathcal{A}$, we only need check it for an algebra (or π-system) that generates $\mathcal{A}$ while showing all elements in possession of such a property are a monotone class (correspondingly, λ-system).

1.3 Measure and Outer Measure

We paid more attention to measures on Euclidean spaces in undergraduate Real Analysis. However, we need abstract measure theories to extend measures to more general spaces.

Definition 6 (Measure)**.** *Given a space X and a σ-algebra $\mathcal{A}$ on it. A **set function** $\mu : \mathcal{A} \to [0, +\infty]$ is a **measure** if $\mu(\varnothing) = 0$ and μ is σ-**additive**, which is*

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

*The triple $(X, \mathcal{A}, \mu)$ is called a **measure space**.*

The following theorem counts when verifying whether a set function is actually a measure.

Theorem 3. *Let $\mathcal{C}$ be a semi-ring and $\mu : \mathcal{C} \to [0, +\infty]$ such that $\mu(\varnothing) = 0$, then μ is σ-additive if and only if μ is **finitely additive** and*

$$\mu\left(E\right) \le \sum_{n=1}^{\infty} \mu(E_n),\ \forall\, E \in \mathcal{C}, E \subset \bigcup_{n=1}^{\infty} E_n.$$

*The last property is **semi-σ-additivity**.*

If we want to extend the domain of μ from $\mathcal{A}$ to X, a relevant concept is taken into consideration.

Definition 7 (Outer measure)**.** *Given a nonempty set X, an **outer measure** on it is a set function $\mu^* : X \to [0, +\infty]$ such that*

1. $\mu^*(\varnothing) = 0$;
2. $E_1 \subset E_2 \Longrightarrow \mu^*(E_1) \leq \mu^*(E_2)$;
3. *For $\{E_n\}_{n=1}^{\infty} \subset X$, we have*
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

The properties of outer measure is similar to that of measure. In fact, we can induce an outer measure from a measure by

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) \middle| E_n \in \mathcal{A}, E \subset \bigcup_{n=1}^{\infty} E_n\right\}.$$

Conversely, we can induce a measure from an outer measure. Given an outer measure μ^*, we say $E \subset X$ is **μ^*-measurable** (or just "measurable") if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c),\ \forall\, A \in X.$$

It is equivalent with

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c),\ \forall\, A \in X,$$

since the other side is trivial. That is to say E could separate any set into two entire pieces. With the help of **Carathéodory criterion** mentioned above, we can restrict an outer measure to a measure.

Remark 2. *There is another concept **premeasure**. It is a set function defined in an algebra instead of a σ-algebra. A premeasure μ_0 is σ-additive and $\mu_0(\varnothing) = 0$, from which we can construct a outer measure, then a measure.*

So far, we have established the basic measure theory, but there is still a tough issue. In fact, we cannot assert that all subsets of a **null set** are measurable. Our common sense tells they are null sets as well. To avoid this barrier, **completion** is necessary.

Definition 8 (Complete measure)**.** *A measure μ is complete if all subsets of null sets are μ-measurable.*

Moreover, there is a unique complete extension $\bar{\mu}$ of μ. The conclusion appeared in our latest assignment.

2 Solutions to Homework

2.1 Exercise 1.2.1

(1)

Proof. Let $\mathcal{R}$ be a ring including $E_1, E_2, \cdots, E_n$, then

$$E_1 \cap E_2 = E_1 \backslash (E_1 \backslash E_2) \in \mathcal{R}.$$

Through induction, it is easy to check the intersection of this n sets belongs to $\mathcal{R}$.

If $\mathcal{R}$ is a σ-ring including $\{E_k\}_{k=1}^{\infty}$, we consider

$$E = \bigcup_{k=1}^{\infty} E_k \in \mathcal{R},$$

then

$$\begin{aligned}\bigcap_{k=1}^{\infty} E_k &= E \backslash \left(E \backslash \bigcap_{k=1}^{\infty} E_k \right) = E \backslash \left(E \cap \left(\bigcap_{k=1}^{\infty} E_k \right)^c \right) \\ &= E \backslash \left(E \cap \left(\bigcup_{k=1}^{\infty} E_k^c \right) \right) = E \backslash \left(\bigcup_{k=1}^{\infty} E \cap E_k^c \right) = E \backslash \left(\bigcup_{k=1}^{\infty} (E \backslash E_k) \right) \in \mathcal{R}.\end{aligned}$$

□

Remark 3. *We wish X were included in $\mathcal{R}$ so that we could take complement and apply De Morgan formula to easily prove the proposition. Even if $X \notin \mathcal{R}$, we can construct a sufficiently large set that takes similar effect with complement. The union of all E_k satisfies our need.*

(3)

Proof. Let

$$Y = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}.$$

It is obvious that Y is closed under complement since $(E^c)^c = E$.

For $\{E_k\}_{k=1}^{\infty} \subset Y$, we consider

$$A = \bigcup_{\substack{k \geq 1 \\ E_k \in \mathcal{R}}} E_k \in \mathcal{R}, \qquad B = \bigcap_{\substack{k \geq 1 \\ E_k^c \in \mathcal{R}}} E_k^c \in \mathcal{R},$$

then

$$\bigcup_{k=1}^{\infty} E_k = \left(\bigcup_{\substack{k \geq 1 \\ E_k \in \mathcal{R}}} E_k \right) \cup \left(\bigcup_{\substack{k \geq 1 \\ E_k^c \in \mathcal{R}}} E_k \right) = A \cup B^c = (A^c \cap B)^c = (B \backslash A)^c.$$

Therefore,

$$B\backslash A \in \mathcal{R} \Longrightarrow \bigcup_{k=1}^{\infty} E_k \in Y$$

□

Remark 4. *Obviously, there are two varieties of sets in Y. Their properties differ, so we must classify them and apply distinct strategies.*

2.2 Exercise 1.3.6

Proof. First, we are going to show that $\overline{\mathcal{M}}$ is a σ-algebra. For $E \in \mathcal{M}, F \subset N \in \mathcal{N}$, we assume that $E \cap N = \varnothing$. Otherwise, we can substitute F, N respectively with $F\backslash E, N\backslash E$. Thus,

$$(E \cup F)^c = ((E \cup N)\backslash(N\backslash F))^c = (E \cup N)^c \cup (N^c \cup F) \in \overline{\mathcal{M}},$$

since $(E \cup N)^c \cup N^c \in \mathcal{M}$.

For such a sequence of sets $\{E_k\}_{k=1}^{\infty}, \{F_k\}_{k=1}^{\infty}, \{N_k\}_{k=1}^{\infty}$, we have

$$\left(\bigcup_{k=1}^{\infty} F_k\right) \subset \left(\bigcup_{k=1}^{\infty} N_k\right) \in \mathcal{N} \Longrightarrow= \left(\bigcup_{k=1}^{\infty} E_k\right) \cup \left(\bigcup_{k=1}^{\infty} F_k\right) \in \overline{\mathcal{M}}.$$

Next, we need to show $\bar{\mu}$ is a complete measure. Obviously,

$$\bar{\mu}(\varnothing) = \mu(\varnothing) = 0.$$

Additionally, for disjoint $\{E_k \cup F_k\}_{k=1}^{\infty}$,

$$\begin{aligned}\bar{\mu}\left(\bigcup_{k=1}^{\infty}(E_k \cup F_k)\right) &= \bar{\mu}\left(\left(\bigcup_{k=1}^{\infty} E_k\right) \cup \left(\bigcup_{k=1}^{\infty} F_k\right)\right) \\ &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k \cup F_k).\end{aligned}$$

Therefore, $\bar{\mu}$ is a measure. Its definition implies completeness since for $F \subset N \in \mathcal{N}$, we have

$$\bar{\mu}(F) = \bar{\mu}(\varnothing \cup F) = \mu(\varnothing) = 0.$$

Finally, the uniqueness is left to be proved. Suppose there is another complete measure $\bar{\mu}'$ extending μ, we have

$$\left.\begin{aligned}\bar{\mu}'(E \cup F) \leq \bar{\mu}'(E \cup N) = \bar{\mu}'(E) \\ \bar{\mu}'(E \cup F) \geq \bar{\mu}'(E \cup \varnothing) = \bar{\mu}'(E)\end{aligned}\right\} \Longrightarrow \bar{\mu}' = \bar{\mu},\ \forall\, E \in \mathcal{M}, F \subset N \in \mathcal{N}.$$

□

Remark 5. *"Complete measure" requests both "completeness" and "measure".*

2.3 Exercise 1.3.8

Proof. By definition,

$$\mu\left(\liminf_{j\to\infty} E_j\right) = \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_j\right) = \mu\left(\lim_{k\to\infty}\bigcap_{j=k}^{\infty} E_j\right)$$

$$= \lim_{k\to\infty}\mu\left(\bigcap_{j=k}^{\infty} E_j\right) \leq \lim_{k\to\infty}\inf_{j\geq k}\mu(E_j) = \liminf_{j\to\infty}\mu(E_j).$$

We can similarly proof the other inequality. □

2.4 Exercise 1.3.10

Proof. Obviously,

$$\mu_E(\varnothing) = \mu(\varnothing\cap E) = \mu(\varnothing) = 0.$$

Let $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ be a sequence of disjoint sets, then $\{A_k \cap E\}_{k=1}^{\infty}$ are disjoint. Thus,

$$\mu_E\left(\bigsqcup_{k=1}^{\infty} A_k\right) = \mu\left(\left(\bigsqcup_{k=1}^{\infty} A_k\right)\cap E\right) = \mu\left(\bigsqcup_{k=1}^{\infty}(A_k\cap E)\right)$$

$$= \sum_{k=1}^{\infty}\mu(A_k\cap E) = \sum_{k=1}^{\infty}\mu_E(A_k).$$

Therefore, μ_E is a measure. □

3 A Glimpse of Hausdorff Measure

3.1 Construction

We don't usually emphasize the dimension when mentioning Lebesgue measures. On some occasions, however, the dimension counts. To provide a universal form of Lebesgue measure in different dimensions, we introduce **Hausdorff measure**.

In fact, there are not only integer dimensions. Cantor ternary set, for example, is a null set under one dimensional Lebesgue measure. However, it includes infinitely many points. So it is not "zero dimensional" as well. An "appropriate" measure on Cantor ternary set seems to have a "fractional" dimension lying between 0 and 1.

To construct n-dimensional Lebesgue measure, we cover a set with countably many n dimensional sets. To measure a d-dimensional set for an arbitrary non-negative real d, we need a “d-dimensional object”. A direct idea is to consider the a one dimensional quantity to the power of d, which is used to cover the original set. Let

$$\mathcal{H}^d_\delta(E) = \inf\left\{ \sum_{k=1}^{\infty} \alpha(d) \left(\frac{diam(E_k)}{2}\right)^d \middle| E \subset \bigcup_{k=1}^{\infty} E_k, diam(E_k) < \delta \right\},$$

where

$$\alpha(d) = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$$

is the “d-dimensional volume of the d-dimensional unit ball”.

There are two variables d and E of this quantity, violating the definition of measure. However, we notice that $\mathcal{H}^d_\delta(E)$ increases as δ tends to 0, since there are fewer choices of $\{E_k\}_{k=1}^{\infty}$.

Definition 9 (Hausdorff outer measure)**.** *In an Euclidean space, the d-dimensional Hausdorff outer measure is defined as*

$$\mathcal{H}^d(E) = \lim_{\delta\to 0^+} \mathcal{H}^d_\delta(E) = \sup_{\delta>0} \mathcal{H}^d_\delta(E).$$

Proof. We need to prove that $\mathcal{H}^d$ is definitely an outer measure.

It is obvious that $\mathcal{H}^d_\delta(\varnothing) = 0$, thus,

$$\mathcal{H}^d(\varnothing) = \lim_{\delta\to 0^+} \mathcal{H}^d_\delta(\varnothing) = 0.$$

Consider $\{E_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$ such that

$$E_k \subset \bigcup_{j=1}^{\infty} A_{kj},\ diam(A_{kj}) < \delta.$$

As a consequence,

$$\mathcal{H}^d_\delta\left(\bigcup_{k=1}^{\infty} E_k\right) \le \alpha(d) \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left(\frac{diam(A_{kj})}{2}\right)^d.$$

Take infimum of $\{A_{kj}\}$ for every fixed k,

$$\mathcal{H}^d_\delta\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \mathcal{H}^d_\delta(E_k) \le \sum_{k=1}^{\infty} \mathcal{H}^d(E_k).$$

Let δ tends to 0, and we obtain the σ-subadditivity.

Ultimately, we have $\mathcal{H}^d(E_1) \le \mathcal{H}^d(E_2)$ for $E_1 \subset E_2$ by definition. □

Now, we can restrict $\mathcal{H}^d$ to measurable sets to obtain Hausdorff measure by Carathéodory criterion.

3.2 Properties

There are some trivial properties of Hausdorff measure.

Theorem 4 (Properties of Hausdorff measure)**.** *Let $\mathcal{H}^d$ be the d-dimensional Hausdorff measure.*

1. *$\mathcal{H}^0$ is* ***counting measure****;*
2. *$\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E)$;*
3. *$\mathcal{H}^d$ is invariant under translation, rotation, and reflection.*

Proof. We only prove the first proposition.

Let $x \in \mathbb{R}^n$, then

$$\alpha(0) = 1 \Longrightarrow \mathcal{H}^0(\{x\}) = 1$$

since we can assume $E_1 = B_\varepsilon(x)$ for small ε and $E_2 = E_3 = \cdots = \varnothing$. □

Next property is significant enough to induce a new concept, which is **Hausdorff dimension**.

Theorem 5 (Uniqueness)**.** *Let $E \subset \mathbb{R}^n$ and $0 \leq d_1 < d_2 < +\infty$, then*

1. *$\mathcal{H}^{d_1}(E) < +\infty \Longrightarrow \mathcal{H}^{d_2} = 0$;*
2. *$\mathcal{H}^{d_2}(E) > 0 \Longrightarrow \mathcal{H}^{d_1} = +\infty$.*

Proof. Abstract $E \subset \mathbb{R}^n$ with $\mathcal{H}^{d_1}_\delta(E) < +\infty$ and $\delta > 0$. By definition, we could find a sequence of sets $\{E\}_{k=1}^\infty$ whose union includes E such that

$$\sum_{k=1}^\infty \alpha(d_1) \left(\frac{diam(E_k)}{2} \right)^{d_1} \leq \mathcal{H}^{d_1}_\delta(E) + 1 \leq \mathcal{H}^{d_1}(E) + 1$$

As a consequence,

$$
\begin{aligned}
\mathcal{H}_\delta^{d_2}(E) &= \sum_{k=1}^{\infty} \alpha(d_2) \left(\frac{diam(E_k)}{2} \right)^{d_2} \\
&= (diam(E_k))^{d_2-d_1} \frac{2^{d_1-d_2}\alpha(d_2)}{\alpha(d_1)} \sum_{k=1}^{\infty} \alpha(d_1) \left(\frac{diam(E_k)}{2} \right)^{d_1} \\
&= \left(\frac{2}{\delta} \right)^{d_1-d_2} \frac{\alpha(d_2)}{\alpha(d_1)} \sum_{k=1}^{\infty} \alpha(d_1) \left(\frac{diam(E_k)}{2} \right)^{d_1} \\
&\leq \left(\frac{\delta}{2} \right)^{d_2-d_1} \frac{\alpha(d_2)}{\alpha(d_1)} \left(\mathcal{H}^{d_1}(E) + 1 \right) \\
&= C\delta^{d_2-d_1}.
\end{aligned}
$$

Let $\delta \to 0$, we obtain the first conclusion. The second proof is similar. □

This theorem implies there is a unique d for fixed E such that $\mathcal{H}^d(E)$ is "neither too big nor too small".

Definition 10 (Hausdorff dimension)**.** *The Hausdorff dimension of a set E is*

$$H_{dim}(E) = \inf\{d \mid \mathcal{H}^d(E) = 0\} = \sup\{d \mid \mathcal{H}^d(E) = +\infty\}.$$

Moreover, E has ***strict Hausdorff dimension*** *d if $\mathcal{H}^d(E) \in (0, +\infty)$.*

3.3 Relation with Lebesgue Measure

n-dimensional Hausdorff measure looks like n-dimensional Lebesgue measure. We are going to show the are definitely identical. First, we need some preparations.

Theorem 6 (Isodiametric inequality)**.** *Let m be then n-dimensional Lebesgue measure, then*

$$m(E) \leq \alpha(n) \left(\frac{diam(E)}{2} \right)^n.$$

This theorem implies that balls have the largest volume for a given diameter. We omit the proof, which is sophisticated. Interested reader could refer to L.Simon. *Introduction to Geometric Measure Theory*-P12 or L.Evans.,R.Gariepy. *Measure Theory and Fine Properties of Functions*-P87.

Theorem 7. *Let m be then n-dimensional Lebesgue measure, then*

$$m(E) = \mathcal{H}^n(E) = \mathcal{H}_\delta^n(E)$$

Proof. Consider

$$E \subset \bigcup_{k=1}^{\infty} E_k.$$

Without loss of generality, we can assume E_k open. Otherwise, we substitute E_k with

$$\tilde{E}_k = \{x \in \mathbb{R} \mid dist(x, E_k) < 2^{-k}\varepsilon\}$$

for sufficiently small ε.

The regularity of Borel sets implies there is a sequence of disjoint closed balls $\{B_{kj}\}_{j=1}^{\infty}$ for fixed k such that

$$\bigcup_{j=1}^{\infty} B_{kj} \subset E_k \text{ and } m\left(E_k \backslash \bigcup_{j=1}^{\infty} B_{kj}\right) = 0.$$

Note that

$$m\left(E_k \backslash \bigcup_{j=1}^{\infty} B_{kj}\right) = 0 \Longrightarrow \mathcal{H}_\delta^n\left(E_k \backslash \bigcup_{j=1}^{\infty} B_{kj}\right) = 0.$$

We have

$$\begin{aligned} \mathcal{H}_\delta^n(E_k) = \mathcal{H}_\delta^n\left(\bigcup_{j=1}^{\infty} B_{kj}\right) &\leq \sum_{j=1}^{\infty} \alpha(n)\left(\frac{diam(B_{kj})}{2}\right)^n \\ &= \sum_{j=1}^{\infty} m(B_{kj}) = m\left(\bigcup_{j=1}^{\infty} B_{kj}\right) = m(E_k) \end{aligned}$$

and thus,

$$\mathcal{H}_\delta^n(E) \leq \mathcal{H}_\delta^n\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mathcal{H}_\delta^n(E_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Taking infimum, we obtain

$$\mathcal{H}_\delta^n(E) \leq m(E).$$

In an effort to prove the reverse inequality, we apply the isodiametric inequality

$$m(E) \leq m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) \leq \sum_{k=1}^{\infty} \alpha(n)\left(\frac{diam(E_k)}{2}\right)^n.$$

Taking infimum again, we reach the conclusion

$$\mathcal{H}^n_\delta(E) = m(E).$$

The other identity comes from taking limit of δ. □

Remark 6. *To specify the dimension, we sometimes substitute dx with $d\mathcal{H}^n$. An example is divergence theorem*

$$\int_\Omega div\, u \, d\mathcal{H}^n = \int_{\partial\Omega} u \, d\mathcal{H}^{n-1}.$$

3.4 Application: Fractals

Here we revisit the dimension issue of Cantor ternary set $\mathcal{C}$.

Figure 1: Cantor ternary set

Given $\delta > 0$, we first choose N so large that the length of each interval $3^{-N} < \delta$. Since the sequence $\{\mathcal{C}_k\}_{k=1}^N$ covers $\mathcal{C}$ and consists of 2^N intervals of diameter $3^{-N} < \delta$, we have

$$\mathcal{H}^d_\delta(\mathcal{C}) \leq 2^N \left(3^{-N}\right)^d.$$

Let $d = \log_3 2$, we have $\mathcal{H}^d_\delta(\mathcal{C}) \leq 1$, thus $H_{dim}(\mathcal{C}) \geq \log_3 2$. And with the help of **Cantor-Lebesgue function**, we can show that $\mathcal{H}^d_\delta(\mathcal{C}) > 0$ (please refer ro E.Stein.*Real Analysis*-P331).

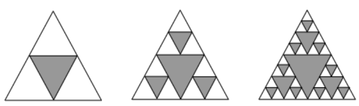

Figure 2: Sierpinski triangle

Similarly, we can construct other kinds of **fractals**, such as Sierpinski triangle and Koch curve, whose Hausdorff dimensions are respectively $\log_2 3$ and $\log_3 4$.

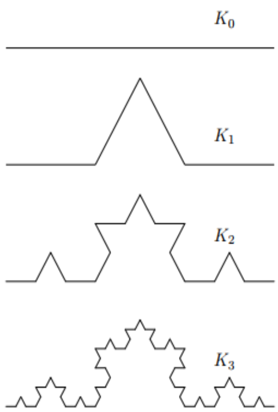

Figure 3: Koch curve

Does there exist a set whose Hausdorff dimension is d for every fixed non-negative d? The answer is “yes”. Interested reader may search the key word “Peter Jones problem”.

Hausdorff measure is the origin of **geometric measure theory**, a popular topic in analysis, such PDEs on irregular domains. Since time is limited, we only gave a brief introduction. There are an enormous quantity of issues in this kingdom, left for you to learn, solve and enjoy.