2.1.1. ⇒ 由于 $\in \mathcal{L}(X,Y)$, T有界, 线性, 结论显然

⇐ $T\{x\in X: \|x\|=1\}$ 有界 ∴ T 有界 □.

2.1.2. (1) "≥" for $\forall x$, $\|x\|\le 1$.

$$\|Ax\| = \|x\|\cdot\|A\tfrac{x}{\|x\|}\| \le \|A\tfrac{x}{\|x\|}\| \le \sup_{\|x\|=1}\|Ax\| = \|A\|$$

take sup for x, $\|x\|\le 1$

$$\sup_{\|x\|\le 1}\|Ax\| \le \|x\|$$

"≤" $$\|x\| = \sup_{\|x\|=1}\|Ax\| \le \sup_{\|x\|\le 1}\|Ax\|.$$

(2) "≥" $$\sup_{\|x\|<1}\|Ax\| \le \sup_{\|x\|\le 1}\|Ax\| = \|A\|.$$

"≤" for $\forall \varepsilon>0$, take x s.t. $\|x\|=1$,

$\|Ax\| \ge \|A\|-\varepsilon$. (we only need to consider $A\ne 0$ otherwise, it is trivial.)

$$\therefore \|A\|-\varepsilon \le \|Ax\| \le \frac{1}{1-\varepsilon}\|A(1-\varepsilon)x\| \le \frac{1}{1-\varepsilon}\sup_{\|x\|<1}\|Ax\|$$

let $\varepsilon\to 0$. $\|A\| \le \sup_{\|x\|<1}\|Ax\|$ □.

2.1.5. $f\ne 0$ ∴ $\exists x$ s.t. $f(x)\ne 0$ ∴ $f\left(\frac{x}{f(x)}\right)=1$ ⇒ $d\ne+\infty$.

for $\forall x$, $\|x\|=1$, $f(x)\ne 0$

we have $f\left(\frac{x}{f(x)}\right)=1$ ⇒ $\|\frac{x}{f(x)}\|\ge d$. $\frac{|f(x)|}{\|x\|}\le\frac{1}{d}$.

i.e. for $\forall \|x\|=1$ $f(x)=0$, or $\frac{|f(x)|}{\|x\|}\le\frac{1}{d}$ ⇒ $\|f\|\le\frac{1}{d}$.

for $\forall\varepsilon>0$, take x s.t. $\|x\|\le d+\varepsilon$, $f(x)=1$.

(since f is bounded, we know $d\ne 0$).

∴ $\|f\| \ge \frac{|f(x)|}{\|x\|} \ge \frac{1}{d+\varepsilon}$ let $\varepsilon\to 0$ $\|f\|\ge\frac{1}{d}$ □.

2.1.6. if $f=0$, it is trivial.

if $f\ne 0$, for $\forall\delta>0$, $\exists x_0$, $\|x_0\|=1$, s.t. $|f(x_0)| \ge \|f\|-\delta$.

WLOG, we assume $f(x_0)\ge\|f\|-\delta$, otherwise we can consider $-x_0$.

∴ let $x_1=\frac{\|f\|}{f(x_0)}x_0$ ∴ $f(x_1)=\|f\|$ $\|x_1\| = \frac{\|f\|}{f(x_0)} \le \frac{\|f\|}{\|f\|-\delta} < 1+\varepsilon$

if δ small enough, □.

2.1.7. (1). for $x,y\in N(T)$ $T(ax+by)=aTx+bTy=0$ ⇒ $ax+by\in N(T)$

∴ $N(T)$ is a linear subspace.

for $x_n\in N(T)$, $x_n\to x_0$. since T is continous $0=Tx_n\to Tx_0$

∴ $Tx_0=0$ ⇒ $x_0\in N(T)$ ∴ $N(T)$ is closed.

(2) NO! Take any Banach space X. and its hamel basis $\{e_\lambda\}_{\lambda\in\Lambda}$.

we choose a sequence $\{e_n\}_{n=1}^{+\infty}$, then let $\{e_\lambda\}_{\lambda\in\Lambda} = \{e_n\}_{n=1}^{+\infty} \amalg \{f_\lambda\}_{\lambda\in\Lambda}$

$\therefore$ we define
$$T: X \to X.\quad e_n \mapsto n e_n,\quad f_\lambda \mapsto f_\lambda$$
$\therefore f$ is linear. $N(T)=0$

but T is not bounded. since $\|T\| \geq \frac{\|Te_n\|}{\|e_n\|} = n$ for $\forall n$.

Rmk: zgg书的答案是不对的

因为 l^1 空间在 l^∞ 范数下不是完备的

例如 $a_n = (1, 2, \cdots, \frac{1}{n}, \cdots)$ 它是Cauchy列. 但不收敛.

(3). $\Rightarrow$ By (1)

$\Leftarrow$ we suppose f is not bounded. $\therefore \exists x_n$. $\|x_n\|=1$. s.t. $|f(x_n)| \geq n$.

WLOG we assume $f(x_n) \geq n$. otherwise, we can consider $e^{i\theta}x_n$.

$\therefore$ we define $y_n = \frac{x_n}{f(x_n)}$ $\therefore f(y_n)=1$. $\|y_n\| \leq \frac{1}{n}$ $\therefore y_n \to 0$.

$\therefore y_n - y_1 \in N(T)$. $\Rightarrow -y_1 \in N(T)$ since $N(T)$ is closed.

However, $f(-y_1) = -1 \neq 0$. Contradiction!

$\therefore f$ is bounded.

HW: 1°. $T: X \to Y$. linear. $\dim X = n < +\infty$.

we choose $\{e_i\}_{i=1}^n$ s.t. $X = \text{span}\{e_1 \cdots e_n\}$

we define $\|x\|_0 = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$ for $x = \sum_{i=1}^n a_i e_i$.

$\therefore \exists C > 1$. s.t. $\frac{1}{C}\|x\| \leq \|x\|_0 \leq C\|x\|$.

we set $M = \sup_{i=1,2,\cdots n} \|Te_i\| < +\infty$

$\therefore$ for $\forall x = \sum_{i=1}^n a_i e_i$ $\|x\|=1$ $\therefore \|x\|_0 = \left(\sum_{i=1}^n a_i^2\right)^{1/2} \leq C$.

$$\|Tx\| = \left\|\sum_{i=1}^n a_i Te_i\right\| \leq M \sum_{i=1}^n |a_i|. \leq M\sqrt{n\sum_{i=1}^n a_i^2} \leq M\sqrt{n}C < +\infty.$$

$\therefore T$ is bounded.

2°. like 2.1.7(2). take Hamel basis $\{e_n\}_{n=1}^{+\infty} \amalg \{f_\lambda\}_{\lambda\in\Lambda}$ of X

and define $T: X \to Y$. where $y \in Y$. $y \neq 0$. □
$$e_n \mapsto n y.\quad f_\lambda \mapsto y.$$

补充：任何线性空间存在 Hamel 基

Pf: 定义 S 为 X 中所有线性无关向量组的全体

并定义偏序关系为包含关系

考虑 $\{A_\alpha\}_{\alpha\in\Lambda}$ 为 S 中的某个全序子集

令 $A=\bigcup\limits_{\alpha\in\Lambda}A_\alpha$ $\therefore$ 对 $\forall$ $x_1\cdots x_n\in A$

$\exists$ $A_1\cdots A_n$ s.t. $x_i\in A_i$. 不妨设 $A_1\subseteq\cdots\subseteq A_n$（由全序）

$\therefore$ $x_1\cdots x_n\in A_n$ $\Rightarrow$ $x_1\cdots x_n$ 线性无关 $\Rightarrow$ A 线性无关，$A\in S$.

$\therefore$ 由 Zorn 引理. S 中存在极大元 M. 下证 M 为 Hamel 基

假设 M 不是 Hamel 基，则 $\exists\, x\in X$. s.t. x 无法被 M 中元素线性表示.

$\therefore$ $M\cup\{x\}$ 线性无关，这与 M 极大矛盾 $\square$

补充：无穷维 B 空间的 Hamel 基一定不可数.

Pf: 假设 X 有可数 Hamel 基 $\{e_i\}_{i=1}^{+\infty}$

令 $X_n=\text{span}\{e_1,\cdots e_n\}$.

$\therefore$ $X=\bigcup\limits_{n=1}^{+\infty}X_n$.

但 X_n 是 X 中的真闭子集，一定是疏集

$\therefore$ $X=\bigcup\limits_{n=1}^{+\infty}X_n$ 为第一纲集，矛盾. $\square$