考试题

1. B^*空间：显然.

完备性：若$\{x^{(m)}\}$为Cauchy列. $\Rightarrow \forall k$. $\{x_k^{(m)}\}$为Cauchy列.

$\therefore \forall k$. $x_k^{(m)} \to x_k$. $\because |x_k^{(m)} - x_k^{(n)}| \le \varepsilon \overset{m\to+\infty}{\Longrightarrow} |x_k^{(n)} - x_k| \le \varepsilon$

$\therefore x^{(m)} \to x$ in ℓ^∞ sense.

最后. for $\forall \varepsilon > 0$. $|x_k| \le |x_k^{(m)} - x_k| + |x_k^{(m)}| \le \varepsilon$.

其中 $\|x^{(m)} - x\| \le \frac{1}{2}\varepsilon$. $|x_k^{(m)}| < \frac{1}{2}\varepsilon$.

2. 由 $C[0,1]$ 在 $L^2[0,1]$ 中稠密 $\therefore C[0,1]^\perp = \{0\}$.

3. 略. 见书本定理1.4.20. B^*空间的有穷维子空间为闭子空间.

4. 考虑 $X = \{x \in \ell^2 \mid x$ 只有有限项不为$0\}$.

$f: X \to \mathbb{R}$.

$(x_1, \cdots x_n \cdots) \longmapsto \sum_{n=1}^{+\infty} \frac{1}{n} x_n$.

By Cauchy. $|f(x)| \le \left(\sum_{n=1}^{+\infty} \frac{1}{n^2}\right)^{1/2} \|x\|_{\ell^2} \le C\|x\|_{\ell^2}$.

但 $\forall y \in X$. $f(x) \ne \langle x, y \rangle$.

5. 只需证明 (1) $M^\perp = \overline{\operatorname{span} M}^\perp$. (2) M为闭子空间时. $(M^\perp)^\perp = M$.

(1) 由 $M \subseteq \operatorname{span}M \Rightarrow (\operatorname{span}M)^\perp \subseteq M^\perp$.

对 $\forall x \in M^\perp$ $\forall y \in \operatorname{span}M$. 设 $y = \sum_{i=1}^n \lambda_i y_i$. $y_i \in M$.

$\therefore \langle x, y \rangle = \sum_{i=1}^n \lambda_i \langle x, y_i \rangle = 0 \Rightarrow x \in \operatorname{span}M^\perp$.

$\therefore M^\perp = (\operatorname{span}M)^\perp$.

$\therefore$ 只需证 $\forall M$. $M^\perp = \overline{M}^\perp$. 由 $M \subseteq \overline{M} \Rightarrow \overline{M}^\perp \subseteq M^\perp$.

对 $\forall x \in M^\perp$ $\forall y \in \overline{M}$. $\exists y_n \in M$. $y_n \to y$.

$\therefore \langle x, y \rangle = \lim_{n\to+\infty} \langle x, y_n \rangle$ (连续性) $= 0$. $\Rightarrow x \in \overline{M}^\perp$.

$\therefore M^\perp = \overline{M}^\perp$ □.

(2) $\forall x \in M$. $\forall y \in M^\perp$. $\langle x, y \rangle = 0 \Rightarrow x \in (M^\perp)^\perp \Rightarrow M \subseteq (M^\perp)^\perp$.

$\forall x \in (M^\perp)^\perp$ 由M为闭子空间, $x = x_1 + x_2$. $x_1 \in M$. $x_2 \in M^\perp$

由 $x \in (M^\perp)^\perp$ $\langle x, x_2 \rangle = 0$ $\because x_1 \in M$. $\therefore \langle x_1, x_2 \rangle = 0$

$\therefore \langle x_2, x_2 \rangle = 0 \Rightarrow x_2 = 0 \Rightarrow x = x_1 \in M$ $\therefore (M^\perp)^\perp \subseteq M$

$\therefore (M^\perp)^\perp = M$ □.

6. $\|\sum_{k=n}^{m} x_k\|^2 = \langle \sum_{k=n}^{m} x_k, \sum_{k=n}^{m} x_k \rangle = \sum_{k=n}^{m} \|x_k\|^2$.

$\therefore$ $\sum_{k=1}^{n} x_k$ 为 Cauchy 列 $\iff$ $\sum_{k=1}^{n} \|x_k\|^2$ 为 Cauchy 列

$\Updownarrow$ $\Updownarrow$

$\sum_{k=1}^{+\infty} x_k$ 收敛 $\sum_{k=1}^{+\infty} \|x_k\|^2 < +\infty$. □

7. 题目改为：E, F 为 X 的闭子空间，F 有限维. 则 $E+F$ 也是闭子空间.

pf 不妨设 $\dim F = 1$. 即 $F = \mathrm{span}\{f\}$, $\|f\| = 1$.

$E+F$ 是子空间显然，下证闭. 若 $f \in E$ 则 $E+F = E$ 显然闭

若 $f \notin E$. $\therefore$ $d = \mathrm{dist}(f, E) > 0$.

考虑 $x_n \to x$, $x_n \in E+F$, 令 $x_n = e_n + \lambda_n f$, $e_n \in E$.

$\therefore$ x_n 为 Cauchy 列.

对 $\forall n, m$ 若 $\lambda_n \neq \lambda_m$, $\|x_n - x_m\| = \|e_n + \lambda_n f - e_m - \lambda_m f\|$

$= \|(\lambda_n - \lambda_m) f - (e_m - e_n)\|$.

$= |\lambda_n - \lambda_m| \, \|f - \frac{e_m - e_n}{\lambda_n - \lambda_m}\| \geq d |\lambda_n - \lambda_m|$

$\therefore$ $|\lambda_n - \lambda_m| \leq \frac{1}{d} \|x_n - x_m\|$. (若 $\lambda_n = \lambda_m$, 显然此式也成立)

$\therefore$ λ_n 为 Cauchy 列 $\Rightarrow$ $\{\lambda_n f\}$ 为 Cauchy 列 $\lambda_n \to \lambda$. $\therefore$ $\lambda_n f \to \lambda f$

$\therefore$ $e_n \to x - \lambda f$ 由 E 闭 $\Rightarrow$ $x - \lambda f \in E$ $\Rightarrow$ $x = x - \lambda f + \lambda f \in E+F$

8. (i). 由 ℓ^2 完备，只需证 A 完全有界. 对 $\forall \varepsilon > 0$, 取 N s.t. $\sum_{k=N+1}^{+\infty} |x_k|^2 < \frac{\varepsilon^2}{100}$.

令 $\tilde{A}_N = \{x = (x_k)_{k=1}^{+\infty} \in \ell^2 : |x_k| \leq \frac{1}{k}, k = 1, 2, \cdots N, x_k = 0 \text{ for } k \geq N+1\}$.

$\therefore$ $\tilde{A}_N$ 与 $\mathbb{R}^N$ 的有界子集等距同构. $\therefore$ 是完全有界的.

$\therefore$ $\exists \{x^1, \cdots x^m\}$ 为 $\tilde{A}_N$ 的有穷 $\frac{\varepsilon}{2}$ 网. 且 $\{x^1 \cdots x^m\} \subseteq A$

$\therefore$ 对 $\forall x \in A$. 令 $y = (x_1, \cdots x_N, 0, \cdots 0 \cdots)$ $\therefore$ $\|x - y\|_{\ell^2} < \frac{\varepsilon}{10}$ (由定义)

且 $y \in \tilde{A}_N$ $\Rightarrow$ $\exists i$ s.t. $\|y - x^i\| < \frac{\varepsilon}{2}$ $\Rightarrow$ $\|x - x^i\| < \varepsilon$

$\therefore$ $\{x^1 \cdots x^m\}$ 为 A 的有穷 ε 网 □

(ii). 由于 $\frac{1}{n} e_n \in A$, 线性无关 $\therefore$ A 不可能含于某个有限维子空间.

(iii). 假设 $x \in A$ 为内点. 即 $\exists r$, $B_r(x) \subseteq A$. 取 k s.t. $\frac{1}{k} < \frac{1}{4} r$.

考虑 $y = x + \frac{3}{k}$ $\therefore$ $y \in B_r(x) \subseteq A$

但 $y_k = x_k + \frac{3}{k} \geq \frac{2}{k}$ $\Rightarrow$ $y \notin A$ 矛盾 $\therefore$ A 不含内点. □

Rmk: 题7中, 有限维条件不可去掉. 即闭子空间 + 闭子空间不一定为闭子空间

考虑 $F: X \to X$ 为有界线性泛函, 但 $\mathrm{Im}F$ 不为闭子空间.

例如 $F: l^2 \to l^2$

$$(x_1, \cdots x_n, \cdots) \mapsto (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots \frac{x_n}{n}, \frac{x_{n+1}}{n+1}, \cdots)$$

显然 F 为有界线性泛函.

但 $(1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, 0, \cdots 0 \cdots) = F(\underbrace{1, 1, \cdots 1}_{n个}, 0, \cdots) \in \mathrm{Im}F$.

$(1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots) \in l^2$, 但 $\notin \mathrm{Im}F$

而 $(1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, 0, \cdots) \longrightarrow (1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots)$ in l^2

$\therefore$ $\mathrm{Im}F$ 不闭.

$\therefore$ 考虑 $E = \{(x, y) \mid x \in X, y = F(x)\}$ 为 $X \times X$ 中的闭集, 因为 F 连续

$F = \{(x, 0) \mid x \in X\}$ 为 $X \times X$ 中的闭集

但 $E + F = X \times \mathrm{Im}F$ 不为闭集 □.

作业题:

2.4.1. (1) $p(2\theta) = p(\theta) = 2p(\theta) \Rightarrow p(\theta) = 0$.

(2) $p(\theta) = p(x + (-x)) \le p(x) + p(-x) \Rightarrow p(-x) \ge -p(x)$.

(3) 若 $x_0 = 0$, 则可任取 $x_1 \neq 0$.

$\therefore$ 只需考虑 $x_0 \neq 0$. 令 $X_0 = \mathrm{span}\{x_0\}$.

令 $f(\lambda x_0) = \lambda p(x_0)$.

$\therefore$ 对 $\lambda \ge 0$, $f(\lambda x_0) = \lambda p(x_0) = p(\lambda x_0)$.

对 $\lambda < 0$, $f(\lambda x_0) = -(-\lambda) p(x_0) = -p(-\lambda x_0) \le p(\lambda x_0)$.

$\therefore$ 由 HBT, 延拓存在 □

2.4.2. 正齐次性: $p(\lambda x) = \limsup\limits_{n \to +\infty} \lambda a_n = \lambda \limsup\limits_{n \to +\infty} a_n = \lambda p(x)$ for $\forall \lambda > 0$.

次可加性: $p(x+y) = \limsup\limits_{n \to +\infty} (a_n + b_n)$ (本题应将 X 改为有界数列全体).

令 $a = \limsup\limits_{n \to +\infty} a_n$. $b = \limsup\limits_{n \to +\infty} b_n$.

对 $\forall c$ 为 $a_n + b_n$ 的极限点, 即 $\exists$ 子列 $a_{n_k} + b_{n_k} \to c$.

取 a_{n_k} 的子列 $a_{n_{k_l}} \to \tilde{a}$ $\therefore \tilde{a} \le a$ 由 $a_{n_{k_l}} + b_{n_{k_l}} \to c$

$\therefore b_{n_{k_l}} \to c - \tilde{a} \Rightarrow c - \tilde{a} \le b$ $\therefore c = \tilde{a} + c - \tilde{a} \le a + b$.

$\therefore$ 对 c 取 sup $\Rightarrow \limsup\limits_{n \to +\infty} (a_n + b_n) \le a + b$ □.

2.4.3. 由 $p(x_0)\neq 0$. p 为半范数 $\Rightarrow x_0 \neq 0$. 令 $X_0=\mathrm{span}\{x_0\}$.

∴ 定义 $f(\lambda x_0)=\lambda$.

∴ $|f(\lambda x_0)| = |\lambda| = \dfrac{p(\lambda x_0)}{p(x_0)}$.

由于 $\dfrac{p(x)}{p(x_0)}$ 同样为半范数 由 HBT. 延拓存在 □.

1.5.1. (1): $\Rightarrow$ $x\in \mathring{E}$ 若 $x=\theta$ $\Rightarrow$ $p(x)=0$.

若 $x\neq 0$. $\exists r>0$ s.t. $B_r(x)\subseteq E$. 即 $x+\frac{r}{2}\frac{x}{|x|}\in E$.

(此不妨设 $r<|x|$).

∴ $p(x)\le \dfrac{1}{1+\frac{r}{2}\frac{1}{|x|}} < 1$.

$\Leftarrow$. 若 $p(x)<1$. 由连续性 $\exists \varepsilon,\delta>0$ s.t. $p(y)<1-\varepsilon$ 对 $\forall y\in B(x,\delta)$.

∴对 $\forall y\in B(x,\delta)$. 取 $0<a_y<1-\frac{1}{2}\varepsilon$.

s.t. $\dfrac{y}{a_y}\in E$ ∵ $0\in E$ $\Rightarrow$ $y\in E$.

∴ $B(x,\delta)\subseteq E$ $\Rightarrow$ $x\in E^\circ$.

(2): $\overline{E^\circ}\subseteq \overline{E}$ 显然.

而 $\overline{E^\circ} = \overline{\{x\,|\,p(x)<1\}} = \{x\,|\,p(x)\le 1\}$ (由 p 的定义).

对 $\forall x\in \overline{E}$ $\exists x_n\to x$. $x_n\in E$ 由 $x_n\in E$ $\Rightarrow$ $p(x_n)\le 1$

∴由连续 $\Rightarrow$ $p(x)\le 1$ $\Rightarrow$ $x\in \overline{E^\circ}$ $\Rightarrow$ $\overline{E}\subseteq \overline{E^\circ}$ □.

2.4.5, 若 $x\in X_0$ 则 $\rho(x,X_0)=0$. 且 $\forall f\in X^*$, $f(X_0)=0$ $\Rightarrow$ $f(x)=0$.

若 $x\notin X_0$ ∴ $d=\rho(x,X_0)>0$.

对 $\forall f\in X^*$ $f(X_0)=0$ $\|f\|=1$.

对 $\forall \varepsilon>0$ 取 $y\in X_0$. $d(x,y)<d+\varepsilon$.

∴ $|f(x)| = |f(y)+f(x-y)| = |f(x-y)| \le \|f\|\|x-y\| < d+\varepsilon$.

令 $\varepsilon\to 0$ $|f(x)|\le d$. $\Rightarrow$ RHS $\le d$. for sup of f.

由定理 2.4.7. $\exists f\in X^*$, $f(X_0)=0$, $\|f\|=1$. $f(x)=d$. $\Rightarrow$ RHS $\ge d$. □.

2.4.6. $\Rightarrow$: $\left|f\left(\sum_{k=1}^n a_k x_k\right)\right| = \left|\sum_{k=1}^n a_k C_k\right| \le \|f\|\left\|\sum_{k=1}^n a_k x_k\right\| \le M\left\|\sum_{k=1}^n a_k x_k\right\|$

$\Leftarrow$: 取 $X_0=\mathrm{span}\{x_1,\cdots x_n\}$ 令 $f: X_0\to \mathbb{K}$.

$\sum_{k=1}^n a_k x_k \mapsto \sum_{k=1}^n a_k C_k$

∴由条件 $\forall x\in X_0$ $|f(x)|\le M\|x\|$ $\Rightarrow$ $\|f\|_{X_0}\le M$ 且 $f(x_k)=C_k$

由 HBT, 延拓存在. □

2.4.7. 令 $X_0=\mathrm{span}\{x_2,\cdots,x_n\}$ 由 $x_1\cdots x_n$ 线性无关 $\Rightarrow x_1\notin X_0 \Rightarrow d(x_1,X_0)>0$

$\therefore \exists f_1\in X^*$, s.t. $f(X_0)=0$, $f(x_1)=1$. By 定理2.4.7. 即 $f_1(x_i)=\delta_{i1}$ □.

补充: 证明: $A\subseteq X$ 的凸包为 A 中任意凸组合的全体

即 $\bigcap\limits_{A\subseteq B凸} B=\left\{\sum\limits_{i=1}^n\lambda_ix_i \;\middle|\; \lambda_i\geq0,\ \sum\limits_{i=1}^n\lambda_i=1,\ x_i\in A\right\}$

pf: 记 RHS $=C$. 对 $x=\sum\limits_{i=1}^n\lambda_ix_i$, $y=\sum\limits_{j=1}^m\mu_jy_j\in C$.

有 $\lambda_i,\mu_j\geq0$, $\sum\limits_{i=1}^n\lambda_i=\sum\limits_{j=1}^m\mu_j=1$. $x_i,y_j\in A$

$\therefore$ 对 $\forall t\in[0,1]$ $tx+(1-t)y=\sum\limits_{i=1}^n t\lambda_ix_i+\sum\limits_{j=1}^m(1-t)\mu_jy_j\in C$

$\therefore$ C 为凸集 $\Rightarrow$ LHS $\subseteq C$.

另一方面, 对 $\forall B$, $B\supseteq A$, B 为凸集, 对 $\forall x_i\in A$, $i=1,\cdots n$. $\forall \sum\limits_{i=1}^n\lambda_i=1$, $\lambda_i\geq0$

有 $\sum\limits_{i=1}^n\lambda_ix_i\in C$ $\Rightarrow C\subseteq B$ $\Rightarrow C\subseteq$ LHS

$\therefore$ LHS $=C$ □.