

PDE Summary

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Abstract

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Reference: *Partial Differential Equations* by Zhou Shulin

1 WAVE EQUATION

1.1 Method of Characteristics

Given a first order linear PDE

$$\begin{cases} \partial_t u + a(x,t)\partial_x u + b(x,t)u = f(x,t) \\ u(x,0) = \Phi(x) \end{cases} \quad (1)$$

Assume $x = x(t)$, while $u(x,t) = u(x(t),t) = U(t)$, and

$$\frac{dU}{dt} = \partial_t u + \partial_x u \cdot x'(t)$$

Let $x'(t) = a(x(t),t)$, consequently

$$\frac{dU}{dt} = b(x(t),t)U(t) + f(x(t),t)$$

Let $x(0) = c$, then

$$U(0) = u(x(0),0) = u(c,0)U(0) = \Phi(c)$$

Solve following first order linear ODEs:

$$\begin{cases} \frac{dx}{dt} = a(x(t),t) \\ x(0) = c \end{cases} \quad (2)$$

$$\begin{cases} \frac{dU}{dt} + b(x(t),t)U(t) = f(x(t),t) \\ U(0) = \Phi(c) \end{cases} \quad (3)$$

Eliminating constant c , we deduce the solution.

1.2 Cauchy Problem

1.2.1 Basic conceptions

Initial value problem (Cauchy problem) :

$$\begin{cases} \partial_t^2 u - \Delta u = f(x, t), x \in \Omega \subset R^n \\ u(x, 0) = \phi(x), x \in \Omega \\ u_t(x, 0) = \psi(x), x \in \Omega \end{cases} \quad (4)$$

With boundary values:

- Dirichlet:

$$u(x, t) = g(t), x \in \partial\Omega \quad (5)$$

- Neumann:

$$\frac{\partial}{\partial \mathbf{n}} u(x, t) = g(t), x \in \partial\Omega \quad (6)$$

- Robin:

$$u(x, t) + \alpha(x, t) \frac{\partial}{\partial \mathbf{n}} u(x, t) = g(t), x \in \partial\Omega, \alpha(x, t) > 0 \quad (7)$$

1.2.2 Decomposition of solution

Theorem 1 (Linear combination) if u_1 satisfies

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = 0 \end{cases} \quad (8)$$

u_2 satisfies

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x, 0) = 0 \\ u_t(x, 0) = \psi(x) \end{cases} \quad (9)$$

u_3 satisfies

$$\begin{cases} \partial_t^2 u - \Delta u = f(x, t) \\ u(x, 0) = 0 \\ u_t(x, 0) = 0 \end{cases} \quad (10)$$

Then $u = u_1 + u_2 + u_3$ is a solution to Cauchy problem (7)

Theorem 2 If $u_2 = M_\psi(x, t)$ is a solution to equation (9), we have

$$u_1 = \partial_t M_\phi \quad (11)$$

$$u_3 = \int_0^t M_{f_\tau}(x, t - \tau) d\tau \quad (12)$$

where $f_\tau = f(x, \tau)$.

Theorem 3 (Fourier transform) *If u satisfies equation*

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \quad (13)$$

We take Fourier transform of x :

$$\begin{cases} \partial_t^2 \hat{u}(\xi, t) - 4\pi^2 |\xi|^2 \hat{u}(\xi, t) = 0 \\ \hat{u}(\xi, 0) = \hat{\phi}(\xi) \\ \hat{u}_t(\xi, 0) = \hat{\psi}(\xi) \end{cases} \quad (14)$$

Solving second order linear equation (14), we deduce the solution to equation (17) by Fourier inversion.

1.3 One Dimension Cauchy Problem

We have the form:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x, t) \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases} \quad (15)$$

As $(\partial_t^2 - \partial_x^2)u = (\partial_t + \partial_x)(\partial_t - \partial_x)u$, we can divide equation(18) into two first order linear PDEs. Applying **method of characteristics** and **theorem 2**, we have:

Theorem 4 (D'Alembert's formula)

$$u(x, t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(y) dy + \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} f(y, \tau) dy d\tau \quad (16)$$

When $f(x, t) = 0$, let

$$F(x) = \frac{1}{2}\phi(x) + \frac{1}{2} \int_0^x \psi(y) dy \quad G(x) = \frac{1}{2}\phi(x) - \frac{1}{2} \int_0^x \psi(y) dy$$

We have

$$u(x, t) = F(x+t) + G(x-t) \quad (17)$$

where $F(x+t)$ is called **leftgoing wave** while $G(x-t)$ **rightgoing wave**.

Theorem 5 *If $\phi(x) \in C^2(\mathbb{R})$, $\psi(x) \in C^1(\mathbb{R})$, then **theorem 4** gives a $u(x, t) \in C^2(\mathbb{R} \times \mathbb{R}_+)$, and u is a solution to equation (15).*

Theorem 6 *When ϕ, ψ, f are all odd/even/periodic functions, u is also a(n) odd/even/periodic function.*

1.4 One Dimension Half-line Problem

Assume u satisfies the following equation (by default, $g(t) = 0$):

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x, t), x > 0, t > 0 \\ u(x, 0) = \phi(x), x \geq 0 \\ u_t(x, 0) = \psi(x), x \geq 0 \\ u(0, t) = g(t), t \geq 0 \end{cases} \quad (18)$$

Prolongate ϕ , ψ and f by odd reflection:

$$\begin{aligned} \bar{\phi}(x) &= \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases} \\ \bar{\psi}(x) &= \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases} \\ \bar{f}(x, t) &= \begin{cases} f(x, t), & x > 0 \\ -f(-x, t), & x < 0 \end{cases} \end{aligned}$$

Let \bar{u} satisfies one dimension equation

$$\begin{cases} \partial_t^2 \bar{u} - \partial_x^2 \bar{u} = \bar{f}(x, t) \\ \bar{u}(x, 0) = \bar{\phi}(x) \\ \bar{u}_t(x, 0) = \bar{\psi}(x) \end{cases}$$

Applying **theorem 4**, we have

$$\bar{u} = \frac{1}{2}(\bar{\phi}(x+t) + \bar{\phi}(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \bar{\psi}(y) dy + \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} \bar{f}(y, \tau) dy d\tau$$

When $x \geq t$,

$$u(x, t) = \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(y) dy + \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} f(y, \tau) dy d\tau \quad (19)$$

When $0 < x < t$,

$$\begin{aligned} u(x, t) &= \frac{1}{2}(\phi(x+t) + \phi(x-t)) + \frac{1}{2} \int_{t-x}^{x+t} \psi(y) dy \\ &\quad + \frac{1}{2} \int_{t-x}^t \int_{x-t+\tau}^{x+t-\tau} f(y, \tau) dy d\tau + \frac{1}{2} \int_0^{t-x} \int_{t-\tau-x}^{x+t-\tau} f(y, \tau) dy d\tau \end{aligned} \quad (20)$$

If u is a solution to equation 22, it satisfies compatibility condition at corner point.

Theorem 7 (compatibility condition)

$$\begin{cases} \phi(0) = \lim_{x \rightarrow 0^+} u(x, 0) = u(0, 0) = \lim_{t \rightarrow 0^+} u(0, t) = g(0) = 0 \\ \psi(0) = \lim_{x \rightarrow 0^+} u_t(x, 0) = u_t(0, 0) = \lim_{t \rightarrow 0^+} u_t(0, t) = g'(0) = 0 \\ \phi''(0) = \lim_{x \rightarrow 0^+} (\partial_x^2 u(x, 0) + f(x, 0)) = \lim_{x \rightarrow 0^+} u_{tt}(x, 0) = u_{tt}(0, 0) = \lim_{t \rightarrow 0^+} u_{tt}(0, t) = g''(0) = 0 \end{cases} \quad (21)$$

Theorem 8 *If $\phi(x) \in C^2(\bar{R}_+)$, $\psi(x) \in C^1(\bar{R}_+)$, $f(x, t) \in C^1(\bar{R}_+ \times \bar{R}_+)$ satisfy compatibility condition, then formula (28) and (29) give a $u(x, t) \in C^2(\bar{R}_+ \times \bar{R}_+)$, and u is a solution to equation (18).*

If $g(t) \neq 0$, let $v(x, t) = u(x, t) - g(t)$, v is a solution to a one dimension half-line problem whose boundary value is 0.

1.5 Multidimensional Wave Equations

1.5.1 transform

Time transform:

$$u(x, t) \rightarrow u(x, t + t_0)$$

Space transform:

$$u(x, t) \rightarrow u(x + x_0, t)$$

Scaling transform:

$$u(x, t) \rightarrow u\left(\frac{x}{\lambda} + \frac{t}{\lambda}\right) = u^\lambda(x, t)$$

Lorentz transform:

$$u(x, t) \rightarrow u\left(x - x_0 + \frac{x_0 - vt}{\sqrt{1 - |v|^2}}, \frac{t - v \cdot x}{\sqrt{1 - |v|^2}}\right)$$

1.5.2 three dimensions

Laplacian operator on a three dimensional sphere:

$$\Delta u = \partial_r^2 u + \frac{2}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^2} u \quad (22)$$

As for Δ_{S^2} , we have

$$\int_{S^2} \Delta_{S^2} u \, d\omega = \int_{S^2} \operatorname{div} \nabla u \, d\omega = \int_{\partial S^2} \frac{\partial u}{\partial \mathbf{n}} \, dS = 0$$

Theorem 9 (sphere average) *Let*

$$\begin{cases} \bar{u}(t, r) = \frac{1}{4\pi} \int_{S^2} u \, d\omega \\ \bar{\phi}(t, r) = \frac{1}{4\pi} \int_{S^2} \phi \, d\omega \\ \bar{\psi}(t, r) = \frac{1}{4\pi} \int_{S^2} \psi \, d\omega \end{cases} \quad (23)$$

Integrate equation (22) on S^2 , and plug Δu into three dimensional equation ($f=0$):

$$\partial_t^2 \bar{u} - (\partial_r^2 \bar{u} + \frac{2}{r} \partial_r \bar{u}) = 0$$

Let $\bar{v} = r\bar{u}$, then \bar{v} satisfies

$$\begin{cases} \partial_t^2 \bar{u} - \partial_x^2 \bar{u} = 0 \\ \bar{v}(r, 0) = r\bar{u}(r, 0) = r\bar{\phi}(r) \\ \bar{v}_t(x, 0) = r(\partial_t \bar{u})(r, 0) = r\bar{\psi}(r) \end{cases} \quad (24)$$

Theorem 10 (Kirchhoff's formula) Through the observation $u(0, t) = \bar{u}(0, t) = \partial_r(r\bar{u}(r, t))|_{r=0}$, we deduce (let $y = x + t\omega$)

$$\begin{aligned} u(x, t) &= \frac{d}{dt} \left(\frac{t}{4\pi} \int_{S^2} \phi(x + t\omega) d\omega \right) + \frac{t}{4\pi} \int_{S^2} \psi(x + t\omega) d\omega \\ &= \frac{d}{dt} \left(\frac{1}{4\pi t} \int_{|y-x|=t} \phi(y) dS(y) \right) + \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y) dS(y) \end{aligned} \quad (25)$$

Theorem 11 If $f \neq 0$, according to **theorem 2**, we have

$$\begin{aligned} u(x, t) &= \frac{d}{dt} \left(\frac{1}{4\pi t} \int_{|y-x|=t} \phi(y) dS(y) \right) + \frac{1}{4\pi t} \int_{|y-x|=t} \psi(y) dS(y) \\ &+ \int_0^t \frac{1}{4\pi(t-\tau)} \int_{|x-y|=t-\tau} f(y, \tau) dy d\tau \end{aligned} \quad (26)$$

1.5.3 two dimensions

Theorem 12 (Poisson's formula) Regarding two dimensional wave equation as a particular three dimensional one as $f = 0$, we have

$$u(x, t) = \frac{d}{dt} \left(\frac{1}{2\pi} \int_{|z-x|\leq t} \frac{\phi(z)}{\sqrt{t^2 - |z-x|^2}} dz \right) + \frac{1}{2\pi} \int_{|z-x|\leq t} \frac{\psi(z)}{\sqrt{t^2 - |z-x|^2}} dz \quad (27)$$

1.5.4 characteristic cone

Theorem 13 The value of $u(x_0, t_0)$ is only concerned with values on $B(x_0, t_0)$. This region is called the **domain of dependence**.

As for $B(x_0, t_0)$, the cone composed by $u(x_0, t_0)$ and $B(x_0, t_0)$ is called the **domain of determinant**.

If the characteristic cone of a point (x_0, t_0) intersects $B(x_0, t_0)$, the set including all such points is called the **domain of influence**

Theorem 14 (support of a function)

$$\text{supp } f = \overline{\{x_0 : f(x_0) \neq 0\}}$$

Theorem 15 (limited velocity of propagation) Abstract $\text{supp } u \subset D$, D is a region. Then $\text{supp } u$ enlarges as t increases, while the velocity of enlarging is constant 1.

1.6 Energy Estimation

1.6.1 energy conservation

Abstract a wave equation without f :

$$\partial_t^2 u - \Delta u = 0 \quad (28)$$

Multiple the equation by $\partial_t u$, we have

$$\begin{aligned} \partial_t u (\partial_t^2 u) &= \frac{1}{2} \partial_t u (\partial_t u)^2 \\ \partial_t u \Delta u &= \sum_{i=1}^n \partial_t u (\partial_{x_i}^2 u) = \sum_{i=1}^n (\partial_{x_i} (\partial_t u \partial_{x_i} u) - \partial_t (\partial_{x_i} u) \partial_{x_i} u) \\ &= \sum_{i=1}^n (\partial_{x_i} (\partial_t u \partial_{x_i} u) - \frac{1}{2} \partial_t u (\partial_{x_i} u)^2) \\ &= \operatorname{div}(\partial_t u \nabla u) - \frac{1}{2} \partial_t |\nabla u|^2 \end{aligned}$$

Therefore, we deduce **energy conservation in differential form**:

$$0 = \frac{1}{2} \partial_t u (\partial_t u)^2 - (\operatorname{div}(u_t \nabla u) - \frac{1}{2} \partial_t |\nabla u|^2) = \partial_t \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} \partial_t |\nabla u|^2 \right) - \operatorname{div}(u_t \nabla u) \quad (29)$$

Assuming u and its derivatives of various orders go to 0, when $|x|$ goes to $+\infty$, in another word:

$$\int_{R^n} \operatorname{div}(\partial_t u \nabla u) = \int_{\partial R^n} \partial_t u \frac{\partial u}{\partial \mathbf{n}} = 0$$

Integrate x in R^n , we deduce **energy conservation in integrated form**:

$$0 = \partial_t \int_{R^n} \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 \right) \quad (30)$$

Let

$$E(t) = \int_{R^n} \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 \right) \quad (31)$$

And $E(t)$ is a constant of variable t , called $E(t)$ the energy of equation (28). As for wave equations in $\Omega \subset R^n$, we can similarly define energy $E(t)$ as

$$E(t) = \int_{\Omega} \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 \right) \quad (32)$$

1.6.2 uniqueness of solution

Assuming u_1 and u_2 are two different solutions to equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x, t), x \in \Omega, t > 0 \\ u(x, 0) = \phi(x), x \in \Omega \\ u_t(x, 0) = \psi(x), x \in \Omega \\ u(0, t)|_{\partial\Omega} = g(x, t), t \geq 0 \end{cases} \quad (33)$$

Then $u = u_1 - u_2$ satisfies

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, x \in \Omega, t > 0 \\ u(x, 0) = 0, x \in \Omega \\ u_t(x, 0) = 0, x \in \Omega \\ u(0, t)|_{\partial\Omega} = 0, t \geq 0 \end{cases} \quad (34)$$

Therefore, $E(t) = E(0) = 0$, $\partial_t u = |\nabla u| = 0$, $x \in \Omega$, so u is constant in Ω .

Let $u \rightarrow \partial\Omega$, with restriction of boundary value, $u = 0$.

1.6.3 stability of solution

L^2 norm:

$$\|f(x)\|_{L^2(\Omega)} = \left(\int_{\Omega} |f(x)|^2 dx \right)^{\frac{1}{2}} \quad (35)$$

$$\|f(x)\|_{L^2(0,T,\Omega)} = \left(\int_0^T \int_{\Omega} |f(x)|^2 dx dt \right)^{\frac{1}{2}} \quad (36)$$

$\forall \varepsilon > 0, \exists \eta > 0$, s.t. two wave equation satisfies

$$\begin{cases} \|\phi_1(x) - \phi_2(x)\|_{L^2(\Omega)} \leq \eta \\ \|\nabla\phi_1(x) - \nabla\phi_2(x)\|_{L^2(\Omega)} \leq \eta \\ \|\psi_1(x) - \psi_2(x)\|_{L^2(\Omega)} \leq \eta \\ \|f_1(x) - f_2(x)\|_{L^2(0,T,\Omega)} \leq \eta \end{cases} \quad (37)$$

Then

$$\begin{cases} \|u_1(x) - u_2(x)\|_{L^2(\Omega)} \leq \varepsilon \\ \|\nabla u_1(x) - \nabla u_2(x)\|_{L^2(\Omega)} \leq \varepsilon \\ \|\partial_t u_1(x) - \partial_t u_2(x)\|_{L^2(\Omega)} \leq \varepsilon \end{cases} \quad (38)$$

Factually, let $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$, $\psi = \psi_1 - \psi_2$, $f = f_1 - f_2$, apply energy estimation,

$$\partial_t \int_{\Omega} \left(\frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 \right) dx = \int_{\Omega} f \partial_t u dx \quad (39)$$

Apply mean inequality to right hand side:

$$\int_{\Omega} f \partial_t u dx \leq \frac{1}{2} \int_{\Omega} (|f|^2 + |\partial_t u|^2) dx \leq \frac{1}{2} \int_{\Omega} |f|^2 dx + E(t)$$

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$$\frac{d}{dt} E(t) \leq \frac{1}{2} \int_{\Omega} |f|^2 dx + E(t) \quad (40)$$

Apply Gronwall's inequality(C_T is a function of just T):

$$\begin{aligned}
E(t) &\leq e^t(E(0) + \frac{1}{2} \int_0^t e^{-\tau} \int_{\Omega} |f|^2 \, dx \, d\tau) \\
&\leq e^T(E(0) + \frac{1}{2} \int_0^T \int_{\Omega} |f|^2 \, dx \, d\tau) \\
&\leq C_T(E(0) + \int_0^T \int_{\Omega} |f|^2 \, dx \, d\tau)
\end{aligned} \tag{41}$$

Moreover, let $g(t) = \int_{\Omega} u^2 \partial_t u \, dx$,

$$\begin{aligned}
g'(t) &= \int_{\Omega} u \partial_t u \, dx \leq \frac{1}{2} \int_{\Omega} \partial_t u^2 \, dx + \frac{1}{2} \int_{\Omega} u^2 \, dx \\
&\leq g(t) + 2E(t) \\
&\leq g(t) + 2C_T(E(0) + \int_0^T \int_{\Omega} |f|^2 \, dx \, d\tau) \\
&\forall 0 \leq t \leq T
\end{aligned} \tag{42}$$

Apply Gronwall's inequality again:

$$g(t) \leq C_T(g(0) + E(0) + \int_0^T \int_{\Omega} |f|^2 \, dx \, d\tau) \tag{43}$$

Equation (41) and (43) result in stability.

1.6.4 limited propagating velocity(proved by energy)

Intergrate equation (29) in circular truncated cone $|x - x_0| \leq R - t$, we have ($e(t) = \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2$ is called **energy density**):

$$\begin{aligned}
0 &= \int_V \partial_t e(u) - \operatorname{div}(u_t \nabla u) \, dx \, dt \\
&= \int_V (\partial_t, \operatorname{div}) \cdot (e(u), -u_t \nabla u) \, dx \, dt \\
&= \int_{\partial V} (\partial_t, \operatorname{div}) \cdot \mathbf{n} \, dx \, dt \\
&= - \int_B e(u)(0) \, dx + \int_T e(u)(t_0) \, dx \\
&\quad + \int_K (e(u), -u_t \nabla u) \cdot \frac{1}{\sqrt{2}} \left(\frac{R-t}{|R-t|}, \frac{x-x_0}{|x-x_0|} \right) \, dS
\end{aligned}$$

After calculation,

$$\int_B e(u)(0) \, dx = \int_T e(u)(t_0) \, dx + \operatorname{Flux}[0, t_0] \tag{44}$$

where

$$\text{Flux}[0, t_0] = \frac{1}{2\sqrt{2}} \int_K (\partial_t u - \frac{x-x_0}{|x-x_0|} \cdot \nabla u)^2 + (|\nabla u|^2 - |\frac{x-x_0}{|x-x_0|} \cdot \nabla u|^2) \, dS \quad (45)$$

That is to say, bottom energy = top energy + laterally wasted energy. Because $T, \text{Flux} \geq 0$, when $B = 0$, we have $T = 0, \text{Flux} = 0$.

1.7 Method of Variable Separation

1.7.1 Sturm-Liouville Boundary Value Problem

Abstract equation:

$$\begin{cases} X'' + \lambda X = 0 \\ -\alpha_1 X'(0) + \beta_1 X(0) = 0, \alpha_1 \geq 0, \beta_1 \geq 0, \alpha_1 + \beta_1 > 0 \\ \alpha_2 X'(l) + \beta_2 X(l) = 0, \alpha_2 \geq 0, \beta_2 \geq 0, \alpha_2 + \beta_2 > 0 \end{cases} \quad (46)$$

λ is an eigenvalue of operator $(-\frac{d^2}{dx^2})$.

Theorem 16 (Sturm-Liouville) *Sturm-Liouville boundary value problem has following properties:*

1. All eigenvalues ≥ 0 . Particularly, when $\beta_1 + \beta_2 > 0$, all eigenvalues > 0 .
2. Eigenfunctions corresponding to different eigenvalues are orthogonal. i.e. $\int_0^l X_\lambda(x) X_\mu(x) \, dx = 0, \forall \lambda \neq \mu$.
3. All eigenvalues make up a monotonic increasing series $\{\lambda_n\}, \lim_{n \rightarrow \infty} \lambda_n = +\infty$.
4. Every function can expand according to system of eigenfunctions. i.e. $f(x) = \sum_{n=1}^{\infty} C_n X_{\lambda_n}(x)$.

1.7.2 Solution to Wave Equations in a Bounded Set

Abstract equation

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x, t), 0 \leq x \leq l, t > 0 \\ u(x, 0) = \phi(x), x \in \Omega, u_t(x, 0) = \psi(x), 0 \leq x \leq l \\ u(0, t) = g_1(t), u(l, t) = g_2(t), t \geq 0 \end{cases} \quad (47)$$

If $f = g_1 = g_2 = 0, f \neq 0$, we claim that

$$u(x, t) = T(t)X(x) \quad (48)$$

Plug into equation (47), we get $T''(t)X(x) - T(t)X''(x) = 0$, i.e. $\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$, which leads to a Sturm-Liouville problem:

$$\begin{cases} X'' + \lambda X = 0 \\ T(t)X(0) = 0, T(t)X(l) = 0, t > 0 \end{cases} \quad (49)$$

Immediately, we notice $X(0) = X(l) = 0$. Through discussion, it is easy to confirm $\lambda > 0$.

Solve second order linear equation $X'' + \lambda X = 0$, and calculate 2 constants with boundary values of PDE (equivalent to primary values of ODE), we have:

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n(x) = \sin\left(\frac{n\pi}{l}x\right), n = 1, 2, \dots \quad (50)$$

Additionally solve the equation of $T(t)$:

$$T_n(t) = C_n \cos\left(\frac{n\pi}{l}t\right) + D_n \sin\left(\frac{n\pi}{l}t\right)$$

Therefore,

$$u_n(x, t) = (C_n \cos\left(\frac{n\pi}{l}t\right) + D_n \sin\left(\frac{n\pi}{l}t\right)) \sin\left(\frac{n\pi}{l}x\right) \quad (51)$$

As solutions' property of linear superposition,

$$u(x, t) = \sum_{n=1}^{\infty} (C_n \cos\left(\frac{n\pi}{l}t\right) + D_n \sin\left(\frac{n\pi}{l}t\right)) \sin\left(\frac{n\pi}{l}x\right) \quad (52)$$

Apply **theorem 16** to ϕ and ψ :

$$\begin{aligned} \phi(x) &= \sum_{n=1}^{\infty} \phi_n \sin\left(\frac{n\pi}{l}x\right) \\ \psi(x) &= \sum_{n=1}^{\infty} \psi_n \sin\left(\frac{n\pi}{l}x\right) \end{aligned}$$

where

$$\begin{aligned} \phi_n &= \frac{\int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx}{\int_0^l \sin^2\left(\frac{n\pi}{l}x\right) dx} = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx \\ \psi_n &= \frac{\int_0^l \psi(x) \sin\left(\frac{n\pi}{l}x\right) dx}{\int_0^l \sin^2\left(\frac{n\pi}{l}x\right) dx} = \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{n\pi}{l}x\right) dx \end{aligned}$$

Consider initial values of u , we deduce:

$$C_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi}{l}x\right) dx, D_n = \frac{2}{n\pi} \int_0^l \psi(x) \sin\left(\frac{n\pi}{l}x\right) dx \quad (53)$$

Combining (52) and (53), equation (47) is solved.

If $f \neq 0$, we can similarly let

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{l}x\right) \quad (54)$$

Plug it into original equation:

$$\begin{aligned}
\sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{l}x\right) &= \partial_t^2 u - \partial_x^2 u \\
&= \sum_{n=1}^{\infty} T_n''(t) X_n(x) - \sum_{n=1}^{\infty} T_n(t) X_n''(x) \\
&= \sum_{n=1}^{\infty} (T_n''(t) + \lambda T_n(t)) X_n(x)
\end{aligned} \tag{55}$$

Compare coefficients,

$$T_n''(t) + \lambda T_n(t) = f_n(t) \tag{56}$$

Considering initial values,

$$T_n(0) = \phi_n, \quad T_n'(0) = \psi_n \tag{57}$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{l}x\right) \tag{58}$$

where

$$T_n(t) = \phi_n \sin\left(\frac{n\pi}{l}x\right) + \frac{l}{n\pi} \psi_n \sin\left(\frac{n\pi}{l}x\right) + \frac{l}{n\pi} \int_0^t f_n(\tau) \sin\left(\frac{n\pi}{l}(t-\tau)\right) d\tau \tag{59}$$

If $g_1, g_2 \neq 0$, let

$$v(x, t) = u(x, t) - \frac{(l-x)g_1(t) + xg_2(t)}{l} \tag{60}$$

Then boundary values of v comes to 0.

Theorem 17 (compatibility condition) *Under the condition $g_1 = g_2 = 0$ and $f = 0$, if ϕ, ψ in region $Q = (0, l) \times (0, +\infty)$ satisfy $\phi(x) \in C^3([0, l])$, $\psi(x) \in C^2([0, l])$, $\phi(0) = \phi(l) = \phi''(0) = \phi''(l) = \psi(0) = \psi(l) = 0$, then $u(x, t) \in C(\bar{Q})$ is a classical solution to equation (47).*

2 POTENTIAL EQUATION

2.1 Harmonic Function

We call

$$-\Delta u = f \quad (61)$$

a **potential equation**. When $f = 0$, it is a **harmonic equation** or **Laplacian equation**. Otherwise, we call it a **Poission's equation**.

2.1.1 mean-value property

If $u : \Omega \rightarrow R$ has continuous second partial derivatives, and $\Delta u = 0$, u is called a **harmonic function**.

Theorem 18 (integrate in polar coordinates) *Abstract* $f : R^n \rightarrow R$,

$$\int_{R^n} f(x) dx = \int_0^\infty \int_{\partial B_r(x_0)} f(y) dS(y) dr = \int_0^\infty r^{n-1} dr \int_{\omega=1} f(x_0+r\omega) dS(\omega) \quad (62)$$

$$\frac{d}{dr} \int_{B_r(x_0)} f(y) dy = \int_{\partial B_r(x_0)} f(y) dS(y) \quad (63)$$

Theorem 19 (mean-value property) *Abstract* $u \in C(\Omega), \Omega \subset R^3$,

$$u(x) = \frac{1}{B_r(x)} \int_{B_r(x)} u(y) dy = \frac{3}{4\pi r^3} B_r(x) \int_{B_r(x)} u(y) \quad (64)$$

$$u(x) = \frac{1}{\partial B_r(x)} \int_{\partial B_r(x)} u(y) dy = \frac{1}{4\pi r^2} \partial B_r(x) \int_{\partial B_r(x)} u(y) \quad (65)$$

Theorem 20 *Two items of mean-value properties are equivalent.*

Theorem 21 u is a harmonic function $\Leftrightarrow u$ has mean-value properties.

2.1.2 value estimation

Theorem 22 (Harnack's inequality) *Abstract* $\Delta u = 0, u(x) \geq 0, \forall x \in \Omega \subset R^n$. \forall connected compact sets $V, \exists C = C(\text{dist}(V, \partial\Omega), n)$, s.t.

$$\sup_V u \leq C \inf_V u \quad (66)$$

Theorem 23 (gradient estimation) *Abstract* $u \in C(\overline{B}_R(x_0))$ is harmonic, then

$$|(\nabla u)(x_0)| \leq \frac{n}{R} \max_{\overline{B}_R(x_0)} |u| \quad (67)$$

if we add a condition $u \geq 0$, there will be a stronger conclusion:

$$|(\nabla u)(x_0)| \leq \frac{n}{R} u(x_0) \quad (68)$$

Theorem 24 (Liouville) *A upper bounded or lower bounded harmonic function in R^n is constant.*

2.2 Fundamental Solution and Green's Function

2.2.1 Fundamental Solution

If $\Delta u = \delta$, u is called a fundamental solution. Particularly, if u is a fundamental solution, $\Delta u = 0, \forall x \neq 0$.

Theorem 25 (radial solution) *If $u = u(r)$, then*

$$\Delta u = \partial_r^2 u + \frac{n-1}{r} \partial_r u + \frac{\Delta_{S^{n-1}} u}{r^2} = \partial_r^2 u + \frac{n-1}{r} \partial_r u = 0$$

let $v(r) = \partial_r u$, we have

$$\partial_r v + \frac{n-1}{r} v = 0$$

Solve the equation, we deduce

$$u(r) = \begin{cases} C_1 r^{-(n-2)} + C_2, & n \geq 3 \\ C_1 \ln r + C_2, & n = 2 \end{cases} \quad (69)$$

Theorem 26 (fundamental solution of Laplacian equation)

$$\Gamma(x) = \begin{cases} \frac{1}{\omega_n |x|^{n-2}}, & n \geq 3 \\ -\frac{1}{2\pi} \ln |x|, & n = 2 \end{cases} \quad (70)$$

where ω_n presents the surface area of an n -dimensional unit sphere. It has a singularity $x = 0$.

Theorem 27 (Green's formula) *Abstract $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$*

$$\int_{\Omega} u \Delta v \, dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \mathbf{n}} \, dS - \int_{\Omega} \nabla u \nabla v \, dx \quad (71)$$

Swapping u, v in Green's formula and subtract the other, we derive

Theorem 28 (Green's second formula)

$$\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \mathbf{n}} - v \frac{\partial u}{\partial \mathbf{n}} \right) \, dS \quad (72)$$

Theorem 29 *Abstract $n = 3$, $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a harmonic function in region Ω , then $\forall x_0 \in \Omega$:*

$$u(x) = \frac{1}{4\pi} \int_{\partial\Omega} \left(-u \frac{\partial}{\partial \mathbf{n}} \left(\frac{1}{|y-x|} \right) + \frac{1}{|y-x|} \frac{\partial u}{\partial \mathbf{n}} \right) \, dS(y) \quad (73)$$

Theorem 30 (Poisson's formula) *Abstract $\Delta u = \Delta g = 0$, and $g|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}$. Apply Green's second formula to u and g in Ω :*

$$\int_{\partial\Omega} \left(u \frac{\partial g}{\partial \mathbf{n}} - g \frac{\partial u}{\partial \mathbf{n}} \right) \, dS = 0$$

Subtract it by **theorem 29**:

$$\begin{aligned} u(x_0) &= \int_{\partial\Omega} \left(u \frac{\partial}{\partial \mathbf{n}} \left(g - \frac{1}{4\pi|x-x_0|} \right) + \left(g - \frac{1}{4\pi|x-x_0|} \right) \frac{\partial u}{\partial \mathbf{n}} \right) dS(x) \\ &= \int_{\partial\Omega} \left(u \frac{\partial}{\partial \mathbf{n}} \left(g - \frac{1}{4\pi|x-x_0|} \right) \right) dS(x) \end{aligned}$$

Let $G(x, x_0) = -g(x) + \frac{1}{4\pi|x-x_0|}$ and u satisfies boundary value $u|_{\partial\Omega} = \phi(x)$. Therefore, we derive a formula present u by only its Dirichlet's boundary value:

$$u(x_0) = - \int_{\partial\Omega} \phi(x) \frac{\partial G}{\partial \mathbf{n}}(x, x_0) dS(x) \quad (74)$$

Theorem 31 (Green's function) *Green's function of operator $-\Delta$ in Ω satisfies*

1. $G(x)$ is second order continuously differentiable and harmonic.
2. $G(x) = 0, \forall x \in \partial\Omega$.
3. $-G(x) + \frac{1}{4\pi|x-x_0|}$ is finite at x_0 , second order continuously differentiable and harmonic everywhere.

Theorem 32 *Green's function satisfies*

$$G(x, x_0) = G(x_0, x) \quad (75)$$

Theorem 33 *Green's function in half-space ($x_3 > 0$):*

$$G(x, x_0) = \frac{1}{4\pi|x-x_0|} - \frac{1}{4\pi|x-x_0^*|} \quad (76)$$

where x_0^* and x_0 are symmetric about plane $x_3 = 0$.

Theorem 34 *Green's function in $B_R(0)$:*

$$G(x, x_0) = \frac{1}{4\pi|x-x_0|} - \frac{R}{4\pi|x - \frac{R^2}{|x_0|}x_0||x_0|} \quad (77)$$

Theorem 35 (Poisson's formula) *Combine equation (74) and (77), we can solve Laplacian equation in $B_R(0)$:*

$$u(x) = \frac{R^2 - |x|^2}{4\pi R} \int_{|y|=R} \frac{\phi(y)}{|y-x|^3} dS(y) \quad (78)$$

Theorem 36 (Harnack's inequality) *Abstract u is harmonic in $B_R(x_0)$ and $u \geq 0$, we have an inequality:*

$$\frac{R}{R+r} \frac{R-r}{R+r} u(x_0) \leq u(x) \leq \frac{R}{R-r} \frac{R+r}{R-r} u(x_0) \quad (79)$$

where $r = |x - x_0| < R$.

2.3 Extremum and Modulus Estimation

Abstract equation

$$\mathcal{L}u = -\Delta u + c(x)u = f(x), c(x) \geq 0, x \in \Omega \quad (80)$$

2.3.1 extremum principle

Theorem 37 (weak maximum principle) *If $c(x) \geq 0$, $f(x) < 0$, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies equation (80), then $u(x)$ doesn't reach its non-negative maximum in Ω , i.e. $u(x)$ reaches its maximum at the boundary.*

Constructing auxiliary function $u_\varepsilon(x) = u(x) + \varepsilon(\text{diam}(\Omega)^2 - |x|^2)$, we can derive following conclusion from former theorem:

Theorem 38 *If $c(x) \geq 0$, $f(x) < 0$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies equation (80) and has positive maximum in $\bar{\Omega}$, then $u(x)$ reach its non-negative maximum of $\bar{\Omega}$ on $\partial\Omega$, i.e.*

$$\max_{x \in \bar{\Omega}} u(x) \leq \max_{x \in \partial\Omega} u^+(x) \quad (81)$$

where $u^+(x) = \max\{u(x), 0\}$.

Constructing auxiliary function $v(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}$, we can derive following conclusion from former theorem:

Theorem 39 (Hopf's lemma) *Abstract $B_R \subset R^n, n = 2, 3$. Bounded function $c(x) \geq 0$, in B_R . If $u \in C^2(B_R) \cap C(\bar{B}_R)$ satisfies:*

1. $\mathcal{L}u \leq 0$
2. $\exists x_0 \in \partial B_R$, s.t. $u(x)$ reaches its non-negative maximum in \bar{B}_R at x_0 and $\forall x \neq x_0, u(x) < u(x_0)$

we have:

$$\frac{\partial u}{\partial \nu} \Big|_{x=x_0} > 0 \quad (82)$$

as long as the angle between ν and \mathbf{n} (the unit outer normal vector of ∂B_R at x_0) $< \frac{\pi}{2}$.

Theorem 40 (strong maximum principle) *Abstract $\Omega \subset R^n$ is a bounded region, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $\mathcal{L}u \geq 0$, reaching its maximum of $\bar{\Omega}$ in Ω , then u is constant in $\bar{\Omega}$.*

Theorem 41 *Abstract $\Omega \subset R^n$ is a bounded region, and $u \in C(\bar{\Omega})$ satisfies mean-value property, then u reaches its maximum and minimum on $\partial\Omega$, unless u is constant.*

This theorem also implies if $-\Delta u \leq 0$ as $x \in \Omega$, u reaches its maximum on $\partial\Omega$.

2.3.2 maximum modulus estimation

Abstract potential equation

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases} \quad (83)$$

Constructing auxiliary function $w = u(x) - (G + \frac{F}{2n}(\text{diam}(\Omega)^2 - |x|^2))$ and $\tilde{w} = -u(x) - (G + \frac{F}{2n}(\text{diam}(\Omega)^2 - |x|^2))$, we can derive a significant conclusion for elliptic equations:

Theorem 42 (maximum modulus estimation) *If u is a solution to equation (83), let $G = \max_{\partial\Omega} |g|$, $F = \max_{\partial\Omega} |f|$, we have:*

$$\max_{\Omega} |u(x)| \leq G + CF \quad (84)$$

where C is a constant concerned with dimension and diameter of Ω .

Theorem 43 (stability of solution) *Abstract equations*

$$\begin{cases} -\Delta u_i = f_i \\ u_i|_{\partial\Omega} = g_i \end{cases} \quad (85)$$

where $i = 1, 2$.

Let $u = u_1 - u_2$, $f = f_1 - f_2$, $g = g_1 - g_2$. Apply **theorem 42**, and we have:

$$\max_{\Omega} |u_1 - u_2| \leq \max_{\partial\Omega} |g_1 - g_2| + C \max_{\partial\Omega} |f_1 - f_2| \quad (86)$$

That is to say, when $g_1 - g_2$ and $f_1 - f_2$ are small enough, $u_1 - u_2$ is arbitrarily small.

Furthermore, if $g_1 = g_2$, $f_1 = f_2$, there must be $u_1 = u_2$, i.e. **uniqueness of solution**.

2.3.3 energy modulus estimation

Theorem 44 (Friedrichs's inequality) *Abstract $u \in C_0^1(\Omega)$, we have:*

$$\int_{\Omega} |u(x)|^2 dx \leq 4d^2 \int_{\Omega} |\nabla u(x)|^2 dx \quad (87)$$

where d is the diameter of Ω

Theorem 45 (energy modulus estimation) *Abstract equation (83) when $g = 0$, multiple the equation by u and integrate its both sides in Ω :*

$$\int_{\Omega} u(-\Delta u) dx = \int_{\Omega} f u dx$$

Integrate by parts, we have

$$-u\Delta u = -u(\nabla \cdot \nabla u) = -\text{div}(u\nabla u) + |\nabla u|^2$$

Therefore,

$$\int_{\Omega} u(-\Delta u) \, dx = - \int_{\partial\Omega} u \frac{\partial u}{\partial \mathbf{n}} \, dS + \int_{\partial\Omega} |\nabla u|^2 \, dx = \int_{\partial\Omega} |\nabla u|^2 \, dx$$

Meanwhile,

$$|fu| \leq \varepsilon |u|^2 + \frac{1}{4\varepsilon} |f|^2$$

Therefore,

$$\int_{\Omega} |\nabla u|^2 \, dx = \left| \int_{\Omega} fu \, dx \right| \leq \int_{\Omega} |fu| \, dx \leq \varepsilon \int_{\Omega} |u|^2 \, dx + \frac{1}{4\varepsilon} \int_{\Omega} |f|^2 \, dx$$

Apply Friedrish's inequality and make ε small enough:

$$\int_{\Omega} |\nabla u|^2 \, dx \leq 4\varepsilon d^2 \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{4\varepsilon} \int_{\Omega} |f|^2 \, dx$$

We deduce:

$$\int_{\Omega} |\nabla u|^2 \, dx \leq C \int_{\Omega} |f|^2 \, dx \quad (88)$$

Additionally, we can apply Friedrish's inequality again:

$$\int_{\Omega} |u|^2 \, dx \leq 4d^2 C \int_{\Omega} |f|^2 \, dx \quad (89)$$

That is to say, both $|u|$ and $|\nabla u|$ are controled by $|f|$.

2.3.4 variable separation

Abstract a Laplacian equation in a boundary region $\Omega = \{(x, y) | x^2 + y^2 = 1\}$

$$\begin{cases} \Delta u = 0, (x, y) \in R^n \\ u = \phi(x), (x, y) \in R^n \end{cases} \quad (90)$$

Let $x = r \cos \theta$, $y = r \sin \theta$, we have:

$$\begin{cases} \Delta u = \partial_r^2 u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_\theta^2 u \\ u|_{r=1} = \phi(\cos \theta, \sin \theta) = \tilde{\phi}(\theta) \end{cases} \quad (91)$$

Plug $u(r, \theta) = R(r)\Theta(\theta)$ into equation (90) and separate variables,

$$-r^2 \frac{R''(r) + \frac{1}{r} R'(r)}{R(r)} = \frac{\Theta(\theta)}{\Theta(\theta)} = -\lambda$$

Therefore,

$$\Theta(\theta) = \begin{cases} C_1 e^{-\sqrt{-\lambda}\theta} + C_2 e^{\sqrt{-\lambda}\theta}, & \lambda > 0 \\ C_1 \theta + C_2, & \lambda = 0 \\ C_1 \cos(\sqrt{\lambda}\theta) + C_2 \sin(\sqrt{\lambda}\theta), & \lambda < 0 \end{cases} \quad (92)$$

Noticing boundary value $\Theta(\theta + 2\pi) + \Theta(\theta)$, there must be $\lambda > 0$. Additionally, $\lambda = n^2$, where $n = 1, 2, \dots$, i.e.

$$\Theta_n(\theta) = C_1 \cos(n\theta) + C_2 \sin(n\theta)$$

As for $R(r)$,

$$r^2 R''(r) + rR'(r) - n^2 R(r) = 0 \quad (93)$$

Let $r = e^t$, then $\partial_t R = r\partial_r R$ and $\partial_t^2 R = r^2 \partial_r^2 R + r\partial_r R$. Plug into equation (93), we deduce an ODE

$$\partial_t^2 R_n - n^2 R_n = 0$$

whose solution is

$$R_n(r) = \begin{cases} C_1 e^{nt} + C_2 e^{-nt} = C_1 r^n + C_2 r^{-n}, & n \geq 1 \\ C_1 t + C_2 = C \ln r + C_2, & n = 0 \end{cases} \quad (94)$$

To make R_n continuously derivative at origin, let

$$R_n(r) = \begin{cases} r^n, & n \geq 1 \\ 1, & n = 0 \end{cases} \quad (95)$$

Now u is presented as

$$u(r, \theta) = \sum_{n=1}^{\infty} r^n (C_n \cos(n\theta) + D_n \sin(n\theta)) + C_0 \quad (96)$$

Consider boundary value:

$$\tilde{\phi}(\theta) = u(1, \theta) = \sum_{n=1}^{\infty} (C_n \cos(n\theta) + D_n \sin(n\theta)) + C_0 \quad (97)$$

Due to orthogonality of trigonometric series, it is easy to determine

$$C_n = \frac{1}{\pi} \int_0^{2\pi} \tilde{\phi}(\theta) \cos(n\theta) \, d\theta \quad (98)$$

$$D_n = \frac{1}{\pi} \int_0^{2\pi} \tilde{\phi}(\theta) \sin(n\theta) \, d\theta \quad (99)$$

$$C_0 = \int_0^{2\pi} \tilde{\phi}(\theta) \, d\theta \quad (100)$$

Plugging them into equation (96), we can present the solution.

3 HEAT EQUATION

3.1 Cauchy Problem of Free Heat Equation

Abstract equation

$$\begin{cases} \partial_t u - \Delta u = f(x, t), x \in R^n, t > 0 \\ u(x, 0) = \phi(x) \end{cases} \quad (101)$$

3.1.1 Fourier transform

Theorem 46 (Fourier transform in R^n) If $f \in L^1(R^n)$, i.e. $\int_{R^n} |f(x)| dx < +\infty$, we define

$$\hat{f}(\xi) = \int_{R^n} f(x) e^{-2\pi i x \cdot \xi} dx \quad (102)$$

Theorem 47 (Schwartz space) We use $\mathcal{S}(R^n)$ to represent Schwartz space:

$$f(x) \in \mathcal{S}(R^n) \Leftrightarrow f \in C^\infty(R^n), \sup \langle x \rangle^\beta |\partial_x^\alpha f| < +\infty$$

Theorem 48 (properties of Fourier transform) Fourier transform has following properties:

1. (translation) If $(\tau_{x_0} f)(x) = f(x - x_0)$:

$$(\widehat{\tau_{x_0} f})(\xi) = e^{-2\pi i x_0 \cdot \xi} \hat{f}(\xi)$$

2. (scaling) If $(S_\lambda f)(x) = f(\lambda x)$:

$$(\widehat{S_\lambda f})(\xi) = \lambda^{-n} \hat{f}(\lambda^{-1} \xi)$$

3. (derivation) If $\alpha = (\alpha_1, \dots, \alpha_n)$, we define $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, then:

$$(\widehat{\partial^\alpha f})(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

$$(\widehat{(-2\pi i \xi)^\alpha f})(\xi) = \partial_\xi^\alpha \hat{f}(\xi)$$

4. (convolution):

$$(\widehat{f * g})(\xi) = \hat{f}(\xi) \hat{g}(\xi)$$

5. (Laplacian):

$$(\widehat{-\Delta f})(\xi) = 4\pi^2 |\xi|^2 \hat{f}(\xi)$$

Theorem 49 (Fourier inversion) If $f \in \mathcal{S}(R^n)$, we define:

$$\check{f}(x) = \int_{R^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi \quad (103)$$

Theorem 50 If $f \in \mathcal{S}(R^n)$:

$$\check{f}(x) = f(x) \quad (104)$$

Here we calculate an significant Fourier transform in advance.
Abstract $f(x) = e^{-x^2}$, let

$$F(\xi) = \hat{f}(\xi) = \int_{R^n} e^{-x^2} e^{-2\pi i x \cdot \xi} dx$$

Derivate F ,

$$\begin{aligned} F'(\xi) &= \int_{R^n} e^{-x^2} (-2\pi i x) e^{-2\pi i x \cdot \xi} dx \\ &= (\pi i) \int_{R^n} (e^{-x^2})' e^{-2\pi i x \cdot \xi} dx \\ &= (\pi i) \hat{f}'(\xi) \\ &= (-w\pi^2 \xi) \hat{F}(\xi) \end{aligned}$$

Solve the ODE above, and notice $F(0) = \int_{R^n} e^{-x^2} dx = \sqrt{\pi}$, we deduce:

$$\hat{f}(\xi) = e^{-\pi^2 \xi^2} \sqrt{\pi}$$

Additionally, former process lead to a deduction:

When $n = 1$,

$$(e^{-4\pi^2 \xi^2 t})(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \quad (105)$$

When $n \geq 2$, consider each component,

$$(e^{-4\pi^2 |\xi|^2 t})(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \quad (106)$$

3.1.2 Solving free heat equation

Apply Fourier transform to equation (101) and let $f = 0$:

$$\begin{cases} \partial_t \hat{u}(\xi) + 4\pi^2 |\xi|^2 \hat{u}(\xi) = 0 \\ \hat{u}(\xi, 0) = \hat{\phi}(\xi) \end{cases} \quad (107)$$

A PDE is transformed into an ODE. Therefore,

$$\hat{u}(\xi, t) = e^{-4\pi |\xi|^2 t} \hat{\phi}(\xi) \quad (108)$$

Apply Fourier inversion:

$$\begin{aligned} u(x, t) &= (e^{-4\pi^2 |\xi|^2 t}) * \phi(x) \\ &= \left(\frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \right) * \phi(x) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} \phi(y) dy \end{aligned} \quad (109)$$

Theorem 51 (heat kernel) Define heat kernel as

$$K(x) = \frac{1}{4(\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4}}, K_t(x) = t^{-\frac{n}{2}} K(t^{-\frac{1}{2}}x) \quad (110)$$

Then, the solution of a free heat equation can be presented as

$$u(x, t) = \int_{R^n} K_t(x-y)\phi(y) \, dy = \int_{R^n} K_t(y)\phi(x-y) \, dy \quad (111)$$

Theorem 52 (properties of heat kernel) Abstract $K(t), t > 0$,

1. $\int_{R^n} K_t(x) \, dx = \int_{R^n} K(x) \, dx = 1$
2. $\int_{R^n} |K_t(x)| \, dx = 1$, i.e. $K_t(x) > 0$
3. $\forall \eta > 0, \int_{|x|>\eta} K_t(x) \, dx \rightarrow 0, t \rightarrow 0^+$

Actually, $\{K_t(x)\}_{t>0}$ is a family of **approximations of the identity**.

Theorem 53 (smoothness of solution) If bounded function $\phi \in C(R^n)$,

$$u(x, t) = K_t(x) * \phi(x) \rightarrow \phi(x), t \rightarrow 0^+ \quad (112)$$

This conclusion also illustrates that $\phi(x)$ is definitely the initial value of $u(x, t)$.

Theorem 54 (properties of solution to free heat equation) If u is a solution to free heat equation when $f = 0$, u has following properties:

1. $u \in C^\infty(R^n), \forall t > 0$
2. $\sup_x |u(x, t)| \leq \sup_x (\phi(x))$
3. Heat equation doesn't satisfy time reversal.
4. Limitless propagating velocity.

Item 4 implies that even if ϕ has an intense support, there's no longer $x_0 \in R^n$ s.t $u(x_0) = 0$ immediately after time starts to pass.

Theorem 55 (solution to nonhomogeneous equation) Apply Fourier transform to equation (101):

$$\begin{cases} \partial_t \hat{u}(\xi) + 4\pi^2 |\xi|^2 \hat{u}(\xi) = \hat{f}(\xi, t) \\ \hat{u}(\xi, 0) = \hat{\phi}(\xi) \end{cases} \quad (113)$$

A PDE is transformed into an ODE once again. Therefore,

$$\hat{u}(\xi, t) = e^{-4\pi |\xi|^2 t} \hat{\phi}(\xi) + \int_0^t e^{-4\pi |\xi|^2 (t-s)} \hat{f}(\xi, s) \, ds \quad (114)$$

Apply Fourier inversion:

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} \phi(y) \, dy + \int_0^t \frac{1}{4(\pi(t-s))^{\frac{n}{2}}} \int_{R^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) \, dy \, ds \quad (115)$$

3.2 Extremum and Modulus Estimation

3.2.1 Dirichlet boundary value problem

Abstract equation

$$\begin{cases} \partial_t u - \Delta u = f(x, t), x \in \Omega, t > 0 \\ u(x, 0) = \phi(x), x \in \Omega \\ u(x, t) = h(x, t), x \in \partial\Omega, t \geq 0 \end{cases} \quad (116)$$

Let $Q_T = \Omega \times (0, T]$, we call $\Gamma_T = \bar{Q}_T \setminus Q_T$ a parabolic boundary.

Theorem 56 (extremum principle) *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u = u_t - u_{xx} = f \leq 0$, $u(x, t)$ reaches its maximum of \bar{Q}_T on parabolic boundary, i.e.*

$$\max_{Q_T} u(x, t) = \max_{\Gamma_T} u(x, t) \quad (117)$$

Naturally, we have a deduction:

Theorem 57 *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u = u_t - u_{xx} = f \geq 0$, $u(x, t)$ reaches its maximum of \bar{Q}_T on parabolic boundary, i.e.*

$$\min_{Q_T} u(x, t) = \min_{\Gamma_T} u(x, t) \quad (118)$$

Theorem 58 (comparison principle) *If $u, v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies equation $\mathcal{L}u \leq \mathcal{L}v$ and $u|_{\Gamma_T} \leq v|_{\Gamma_T}$, we always have $u(x, t) \leq v(x, t)$ in \bar{Q}_T .*

Constructing auxiliary function $v = Ft + B - u$, we can derive maximum modulus estimation in comparison with that of elliptic equations:

Theorem 59 (maximum modulus estimation) *If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a solution to equation:*

$$\begin{cases} \mathcal{L}u = f, x \in (0, l) \times t > 0 \\ u|_{t=0} = \phi(x), x \in [0, l] \\ u|_{x=0} = g_1(t), u|_{x=l} = g_2(t) \end{cases} \quad (119)$$

let

$$F = \max_{Q_T} |f|, \quad B = \max\left\{ \max_{x \in [0, l]} |\phi|, \max_{0 < t \leq T} |g_1(t)|, \max_{0 < t \leq T} |g_2(t)| \right\}$$

and we have a conclusion:

$$\max_{Q_T} |u(x, t)| \leq FT + B \quad (120)$$

3.2.2 Robin and Neumann boundary value problem

Theorem 60 (uniqueness of solution to Robin boundary value problem)

The solution to equation

$$\begin{cases} \partial_t - \partial_x^2 = f(x, t), x \in (0, l) \times t > 0 \\ u(x, 0) = \phi(x), x \in [0, l] \\ u(0, t) = g_1(x), (u_x + hu)(l, t) = g_2(x), h > 0 \end{cases} \quad (121)$$

is unique.

Constructing auxiliary function $\tilde{u} = (-x + l + 1)u$ and $v = e^{-\lambda t}\tilde{u}$, we can prove maximum principle of u . Imitating the proof of Robin boundary value problem, we deduce:

Theorem 61 (uniqueness of solution to Neumann boundary value problem)

The solution to equation

$$\begin{cases} \partial_t - \partial_x^2 = f(x, t), x \in (0, l) \times t > 0 \\ u(x, 0) = \phi(x), x \in [0, l] \\ u(0, t) = g_1(x), u_x(l, t) = g_2(x) \end{cases} \quad (122)$$

is unique.

3.2.3 modulus estimation

Considering $\omega(x, t) = Ft + \Phi + v_L(x, t) \pm u(x, t)$, where $v_L(x, t) = \frac{M}{L^2}(x^2 + 2t)$, we deduce a similar conclusion to **theorem 59**:

Theorem 62 (maximum modulus estimation in R) If $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ is a bounded solution to equation:

$$\begin{cases} \mathcal{L} = f, x \in R, 0 < t \leq T \\ u|_{t=0} = \phi(x), x \in R \end{cases} \quad (123)$$

let

$$F = \sup_{\bar{Q}_T} |f|, \Phi = \sup_{x \in R} |\phi|, M = \sup_{\bar{Q}_T} |u| \quad (124)$$

and we have a conclusion:

$$M \leq FT + \Phi \quad (125)$$

Theorem 63 (energy modulus estimation) If $u \in C^{1,0}(\bar{Q}_T) \cap C^{2,1}(\bar{Q}_T)$ is a solution to equation

$$\begin{cases} \partial_t u - \partial_x^2 = f, x \in (0, l) \times (0, T] \\ u|_{t=0} = \phi(x), x \in [0, l] \\ u|_{x=0} = u|_{x=l} = 0 \end{cases} \quad (126)$$

multiple the equation by u , we have

$$\frac{1}{2} \frac{\partial}{\partial t} u^2 - \partial_x(uu_x) + u_x^2 = fu$$

Integrate two sides on $[0, l]$ and apply mean value inequality:

$$\frac{1}{2} \int_0^l \frac{\partial}{\partial t} u^2 \, dx \leq \frac{1}{2} \int_0^l \frac{\partial}{\partial t} u^2 \, dx + \int_0^l u_x^2 \, dx \leq \frac{1}{2} \int_0^l f^2 \, dx + \frac{1}{2} \int_0^l u^2 \, dx$$

Apply Gronwall's inequality to $y(t) = \int_0^l u^2 \, dx$,

$$y(t) \leq e^t y(0) + e^t \int_0^t \int_0^l f^2(x, s) \, dx \, ds$$

Additionally,

$$\int_0^l u(x, t)^2 \, dx \leq e^T \left(\int_0^l \phi(x)^2 \, dx + \int_0^T \int_0^l f^2(x, s) \, dx \, ds \right)$$

Therefore,

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \int_0^l u^2 \, dx + \int_0^l u_x^2 \, dx &\leq \frac{1}{2} \int_0^l f^2 \, dx + \frac{1}{2} \int_0^l u^2 \, dx \\ &\leq \frac{1}{2} \int_0^l f^2 \, dx + e^T \left(\int_0^l \phi(x)^2 \, dx + \int_0^T \int_0^l f^2(x, s) \, dx \, ds \right) \end{aligned}$$

Integrate two sides on $[0, t]$,

$$\begin{aligned} &\frac{1}{2} \int_0^l u^2(x, t) \, dx + \int_0^t \int_0^l u_x^2(x, s) \, dx \, dt \\ &\leq \int_0^l \phi(x)^2 \, dx + \frac{1}{2} \int_0^t \int_0^l f^2(x, s) \, dx \, ds \\ &\quad + (e^t - 1) \left(\int_0^l \phi(x)^2 \, dx + \int_0^T \int_0^l f^2(x, s) \, dx \, ds \right) \\ &\leq \frac{1}{2} e^T \left(\int_0^l \phi(x)^2 \, dx + \int_0^T \int_0^l f^2(x, s) \, dx \, ds \right) \end{aligned}$$

Due to arbitrariness of t , we have

$$\sup_{0 \leq t \leq T} \int_0^l u^2(x, t) \, dx + 2 \int_0^T \int_0^l u_x(x, t) \, dx \, dt \leq M \left(\int_0^l \phi^2 \, dx + \int_0^T \int_0^l f^2(x, t) \, dx \, dt \right) \quad (127)$$

3.2.4 variable separation

Abstract equation:

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, x \in (0, l) \times (0, T] \\ u(x, 0) = \phi(x), x \in [0, l] \\ u(0, t) = 0, u_x(l, t) + hu(l, t) = 0, h > 0 \end{cases} \quad (128)$$

Plug $u(x, t) = T(t)X(x)$ into the equation,

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad (129)$$

New boundary values are

$$T(t)X(0) = 0, \quad T(t)X'(l) + hT(t)X(l) = 0$$

i.e.

$$X(0) = 0, \quad X'(l) + hX(l) = 0$$

Solving equation (129), we deduce

$$X(x) = \begin{cases} C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}, & \lambda < 0 \\ C_1 x + C_2, & \lambda = 0 \\ C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) & \lambda > 0 \end{cases} \quad (130)$$

To satisfy boundary values, there must be $C_1 = 0$, $\lambda > 0$. Moreover,

$$\tan(\sqrt{\lambda}l) = -\frac{\sqrt{\lambda}}{h} \quad (131)$$

Analysing the graph, we figure out a increasing positive series of $\{\lambda_n\}$ satisfying equation (130). Let $X_n(x) = \sin(\sqrt{\lambda_n}x)$, then $T(t) = Ae^{-\lambda_n t}$ according to former equations, We have

$$u(x, t) = \sum_{n=1}^{\infty} Ae^{-\lambda_n t} \sin(\sqrt{\lambda_n}x) \quad (132)$$

Expand initial value $\phi(x)$:

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\sqrt{\lambda_n}x)$$

where

$$\phi_n = \frac{\int_0^l \phi(x) \sin(\sqrt{\lambda_n}x) dx}{\int_0^l \sin^2(\sqrt{\lambda_n}x) dx} \quad (133)$$

Compare parameters:

$$A_n = \phi_n e^{\lambda_n t}$$

Ultimately, we deduce

$$u(x, t) = \sum_{n=1}^{\infty} \phi_n \sin(\sqrt{\lambda_n}x) e^{-\lambda_n t} \quad (134)$$