Notes of Complex Analysis (H)

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Abstract

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1 PRELIMINARIES

1.1 Complex Numbers

1.1.1 General form

Definition 1.1 (Complex field)

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$$

Operation

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$
$$(a + b\mathrm{i})(c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

Definition 1.2

Real part: $Re(z) = a$

Imaginary part: $Im(z) = b$

Modulus: $|z| = \sqrt{a^2 + b^2}$

Conjugate: $\bar{z} = a - bi$

Properties

$$
\begin{aligned}
&|z| = z\bar{z}, \quad \frac{1}{z} = \frac{\bar{z}}{|z|}\,(z \neq 0)\\
&\mathrm{Re}(z) = \frac{z+\bar{z}}{2}, \quad \mathrm{Im}(z) = \frac{z-\bar{z}}{2\mathrm{i}}\\
&\overline{z+w} = \bar{z}+\bar{w}, \quad \overline{zw} = \bar{z}\bar{w}\\
&|zw| = |z||w|, \quad |\frac{z}{w}| = \frac{|z|}{|w|}\,(w \neq 0)
\end{aligned}
$$

Inequalities

$$
\begin{aligned}
&\mathrm{Re}(z) \leq |z|, \quad \mathrm{Im}(z) \leq |z|\\
&|z+w| \leq |z|+|w|, \quad |z-w| = ||z|-|w||
\end{aligned}
$$

1.1.2 Triangular form

Definition 1.3 *Let* $r = |z|, \theta = \arctan\frac{b}{a} = \arg(z)$*, meaning* $a = r\cos\theta, b = r\sin\theta$*:*

$$z = r(\cos\theta + i\sin\theta)$$

Therefore, we can define the ***argument*** *of a complex number:*

$$Arg(z) = \{\theta + 2k\pi | k \in \mathbb{Z}\}$$

Fixing an interval measuring 2π *such as* $[0, 2\pi)$*, we ulteriorly define* ***argument principal value****:*

$$\arg(z) = \theta$$

Operation Let $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_2), z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_2)$:

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2(\cos(\theta_1+\theta_2) + \mathrm{i}\sin(\theta_1+\theta_2))\\
\frac{z_1}{z_2} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + \mathrm{i}\sin(\theta_1-\theta_2))
\end{aligned}
$$

Properties

$$
\begin{aligned}
|z_1 z_2| &= r_1 r_2\\
\mathrm{Arg}(z_1 \pm z_2) &= \mathrm{Arg}(z_1) \pm \mathrm{Arg}(z_2)
\end{aligned}
$$

Theorem 1.1 (De Moivre's formula)

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

Theorem 1.2 (Euler's formula)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1.2 Complex Plane

Definition 1.4 (Extended complex plane)

$$\bar{\mathbb{C}} = \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$$

Definition 1.5 (General circle) *Let $A, C \in \mathbb{R}, B \in \mathbb{C}, |B|^2 - AC > 0$, and a general circle is presented in form of*

$$Az\bar{z} + \bar{B}z + B\bar{z} + C = 0$$

When $A = 0$, it is a common circle; when $A \neq 0$, it is a line.

Stereographic projection Regarding $\mathbb{C}$ as xOy plane, there is a one-to-one correspondence between $\bar{\mathbb{C}}$ and sphere $S_2 : x^2 + y^2 + z^2 = 1$. Specifically speaking, $N = (0, 0, 1)$, P and z are collinear.

Given z on $\bar{\mathbb{C}}$,

$$P = (\frac{z + \bar{z}}{|z|^2 + 1}, \frac{z - \bar{z}}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1})$$

Given $P = (x_1, x_2, x_3)$ on S_2,

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

Distance between two complex number

$$d(z, w) = |P - Q| = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$

Such definition guarantees boundedness, and when $w \to \infty$,

$$d(z, w) = \frac{2}{\sqrt{1 + |z|^2}}$$

Theorem 1.3 *Circles on S_2 corresponds to general circles on $\bar{\mathbb{C}}$ through stereographic projection one by one.*

1.3 Analytic Properties of Complex Numbers

Definition 1.6 (Limit of complex series) *A complex series $\{z_n\}_{n=1}^{\infty}$ converge to w, i.e*

$$\lim_{n\to\infty} z_n = w$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all n

$$n > N \Longrightarrow |z_n - w| < \varepsilon$$

Theorem 1.4

$$\begin{aligned}&\lim_{n\to\infty} z_n = w\\ \Longleftrightarrow &\lim_{n\to\infty} |z_n - w| = 0\\ \Longleftrightarrow &\begin{cases}\lim\limits_{n\to\infty} Re(z_n) = Re(w)\\ \lim\limits_{n\to\infty} Im(z_n) = Im(w)\end{cases}\end{aligned}$$

Definition 1.7 (Continuity) *$f(z)$ is continuous at the point z_0 if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all z*

$$|z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \varepsilon$$

Definition 1.8 (Derivative) *$f(x)$ is derivable at the point z_0 if limit*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and we denote

$$f'(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Definition 1.9 (Differentiability) *$f(x)$ is differentiable at the point z_0 if*

$$f(z_0 + \Delta z) - f(z_0) = A(z_0)\Delta z + \rho(\Delta z)$$

where

$$\lim_{\Delta z\to z_0} \frac{\rho(\Delta z)}{\Delta z} = 0$$

In particular, differentiability is **equivalent** to derivability in the field of complex variables functions.

Definition 1.10 (Holomorphism) *Abstract complex variables function $f(z)$*

- *$f(z)$ is holomorphic in the region D if $f(z)$ is everywhere derivable in D, denoted as $f(z) \in H(D)$.*
- *$f(z)$ is holomorphic at the point z_0 if $f(z)$ is derivable in a neighbourhood of z_0.*

2 HOLOMORPHIC FUNCTIONS

2.1 Properties of Holomorphic Functions

Expression A complex variables function $f(z)$ can be uniquely expressed as

$$f(z) = u(x, y) + \mathrm{i}\, v(x, y)$$

where $z = x + \in y$ and u, v are real-valued functions.

Definition 2.1 (Real differentiability) *For $f(z) = u(x, y) + iv(x, y)$, $f(z)$ is real differentiable at $z_0 = x_0 + iy_0$, if u and v are differentiable at (x_0, y_0).*

Theorem 2.1 *$f(z)$ is real differentiable at z_0 if and only if*

$$f(z_0 + \Delta z) - f(z_0) = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\Delta \bar{z} + \rho(|\Delta z|), \ \lim_{\Delta z \to 0} \frac{\rho(\Delta z)}{\Delta z} = 0$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Cauchy-Riemman equations Noticing

$$0 = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y})(u + \mathrm{i}\, v) = \frac{1}{2}(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + \frac{\mathrm{i}}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})$$

we call equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

Cauchy-Riemman equations.

Theorem 2.2 (Determination of differentiability) *$f(z)$ is differentiable at z_0 if and only if $f(z)$ is real differentiable at z_0 and satisfies Cauchy-Riemman equations.*

Differential properties Let $f, g \in H(D)$:

$$\begin{aligned} & f \pm g \in H(D) && (f \pm g)' = f' \pm g' \\ & fg \in H(D) && (fg)' = f'g + fg' \\ & \frac{f}{g} \in H(D) && (\frac{f}{g})' = \frac{f'g - fg'}{g^2} \ (g(z) \neq 0) \end{aligned}$$

Chain rule Let $f \in H(G), g \in H(D), f(G) \in D$, then $\phi(z) = f(g(z)) \in G$, and

$$\phi'(z) = g'(f(z))f'(z)$$

2.2 Harmonic Functions

Definition 2.2 (Complex Laplacian operator)

$$\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}}$$

Theorem 2.3

$$f = u + iv \in H(D) \Longrightarrow \Delta u = \Delta v = 0$$

Generally speaking, holomorphism implies harmonicity.

Definition 2.3 (Conjugate harmonic functions) *Harmonic functions f, g defined in D are conjugate harmonic functions if they satisfy* ***Cauchy-Riemann equations****.*

Existence of conjugate harmonic functions Given a simply connected region D and a harmonic function u in D, the conjugate harmonic function of u, v exists, and $f = u + \mathrm{i}\, v \in H(D)$.

In fact,

$$v(x, y) = \int_{(x_0, y_0)}^{(x,y)} -\frac{\partial u}{\partial y}\,\mathrm{d}x + \frac{\partial u}{\partial x}\,\mathrm{d}y$$

2.3 Geometric Properties of Derivative

Definition 2.4 (Smoothness of a curve) *Let $\gamma : [a, b] \to \mathbb{C}, t \mapsto x(t) + iy(t)$*

- *γ is* ***smooth*** *in $[a, b]$ if $\gamma'(t)$ exists in $[a, b]$.*
- *γ is* ***piecewise smooth*** *in $[a, b]$ if $\exists\, a = t_0 < t_1 < \cdots < t_n = b$, $\gamma(t)$ is smooth in $[t_{i-1}, t_i]$.*

Definition 2.5 (Tangent of a curve) *For curve $\gamma(t)$, the angle between its tangent and positive z-axis at z_0 is $Arg(\gamma'(z_0))$.*

Definition 2.6 (Conformal mapping) *Function f is* ***conformal****, if the angle between 2 curves equals that between their images under f.*

Theorem 2.4 *Conformal functions have following properties:*

- *$f \in H(\Omega), f'(z_0) \neq 0 \Longrightarrow f$ is conformal in the neighberhood of z_0.*
- *$f \in H(\Omega), f'(z_0) = 0 \Longrightarrow f$ is* ***never*** *conformal at z_0.*
- *$f \in H(\Omega), f$ is a bijection from Ω to $D, f(z) \neq 0 \Longrightarrow f$ is conformal in Ω.*
- *$f \in H(\Omega), f$ is conformal in $\Omega \Longrightarrow f'(z) \neq 0$.*

Definition 2.7 (Modulus of derivative) *$|f'(z_0)|$ indicates the ratio of scaling of f at z_0:*

$$|f'(z_0)| = \lim_{z \to z_0} |\frac{f(z) - f(z_0)}{z - z_0}| = \lim_{z \to z_0} \frac{|w - w_0|}{|z - z_0|}$$

2.4 Primary Complex Variables Functions

2.4.1 Exponential function

Definition 2.8 (Exponential function)

$$e^z = e^x(\cos y + i \sin y)$$

where $z = x + iy$.

Properties Exponential function has following properties:

1. $|e^z| = e^x|\cos y + \mathrm{i}\sin y| = e^x > 0$.
2. $e^{z_1+z_2} = e^{z_1}e^{z_2}$.
3. $e^z = e^{z+2k\pi\mathrm{i}}, k \in \mathbb{Z}$.
4. $e^z \in H(\mathbb{C})$.

Definition 2.9 (Univalent domain) *A function satisfying*

$$z_1 \neq z_2 \Longrightarrow f(z_1) \neq f(z_2)$$

*is called a **univalent function**.*

*Additionally, if $f(z)$ is univalent in region D, D is called a **univalent domain** of $f(z)$.*

Univalent domains of exponential function

$$e^{z_1} = e^{z_2} \Longleftrightarrow z_1 - z_2 \neq 2k\pi$$

Theorem 2.5 *Abstract $f(z) = e^z$ and zonal region*

$$D = \{z|Im(z) \in (0, h)\},\ \ h < 2\pi$$

D is a univalent domain of $f(z)$, and

$$f : D \mapsto \Omega \tag{1}$$

where

$$\Omega = \{re^{i\theta}|r > 0, 0 < \theta < h\}$$

2.4.2 Logarithmic function

Definition 2.10 (Logarithmic function)

$$Logz = \log|z| + i\,Arg\,z$$

It is a multivalue function.

Single valued branch The k-th single valued branch of Log$z$ is $(\text{Log}z)_k$, i.e.

$$(\text{Log}z)_k = \log|z| + \text{i} \arg z + 2k\pi\text{i}$$

Without confusion, it is denoted as $\log z$.

Properties A single valued branch of a logarithmic function has following properties:

1. Domain of definition: $\mathbb{C}\backslash\{0\}$.
2. Not continous in $(0, +\infty) \subset \mathbb{C}\backslash\{0\}$.

Definition 2.11 (Variation of multivalue function) *Abstract $F(z)$ defined in Ω, $z_0 \in \Omega$ and its initial value $f(z_0)$. When z goes to z_0 continuously along curve $C \subset \Omega$, $f(z)$ also goes to a well-determined value $f(z_0)$. Here we call define the* ***variation of*** *$F(z)$* ***along*** *C, and denote*

$$\Delta_C F(z) = f(z) - f(z_0)$$

Definition 2.12 (Single valued domain) *Ω is a* ***single valued domain*** *of $F(z)$, if $\Delta_C F(z)$ depends on z and z_0, rather than the selection of C. Then $F(z)$ has a* ***single valued branch*** *in Ω.*

Theorem 2.6 *Ω is a single valued domain of $F(z)$ if and only if*

$$\Delta_C F(z) = 0$$

for all simple closed curves in Ω.

Definition 2.13 (Branch point) *A point z_0 is a* ***branch point*** *if it satisfies following requirements:*

1. *$\exists\, r > 0$,s.t. $B_r(z_0) \subset \Omega$ and $F(z)$ is definable in $\check{B}_r(z_0)$.*
2. *Every point in $\check{B}_r(z_0)$ is an ordinary point, i.e. every point has a neighhood, in which $F(z)$ has a single valued branch.*
3. *An arbitrarily small deleted neighborhood of z_0 has a simple closed curve C surrounding z_0, s.t. $\Delta_C F(z) \neq 0$.*

2.4.3 Power function

Definition 2.14 (Power function)

$$f(z) = z^\alpha = e^{\alpha \log z},\ \alpha \in \mathbb{C}$$

Several common cases According to α, power functions are classified into several cases:

- $\alpha = n \in \mathbb{N}$:
$$f(z) = z^n \in H(\mathbb{C})$$
- $\alpha = \frac{1}{n}, n \in \mathbb{N}$:
$$f(z) = z^{\frac{1}{n}} = e^{\frac{1}{n}\log|z|} e^{\mathrm{i}\,\frac{1}{n}\mathrm{Arg}(z)} = |z|^{\frac{1}{n}} e^{\mathrm{i}\,\frac{1}{n}(\arg(z)+2k\pi)}$$
- $\alpha = a + b\mathrm{i}, a, b \in \mathbb{R}$:
$$f(z) = e^{(a+b\mathrm{i})(\log|z|+\mathrm{i}\,\mathrm{Arg}(z))} = e^{a\log|z|-b\mathrm{Arg}(z)} e^{\mathrm{i}(b\log|z|+a\mathrm{Arg}(z))}$$

Classified discussion of value Let $\alpha = a + b\mathrm{i}, a, b \in \mathbb{R}$:

- $b = 0, a = n$:
$$f(z) = z^n$$
is a single valued function.
- $b = 0, a = \frac{p}{q}$:
$$f(z) = z^{\frac{p}{q}} = e^{\frac{p}{q}\log|z|} e^{\mathrm{i}\,\frac{p}{q}\mathrm{Arg}(z)} = |z|^{\frac{p}{q}} e^{\mathrm{i}\,\frac{p}{q}(\arg(z)+2k\pi)}$$
is a q-valued function.
- $b = 0, a \in \mathbb{R}\backslash\mathbb{Q}$:
$$f(z) = |z|^a e^{\mathrm{i}\,a(\arg(z)+2k\pi)}$$
is an infinitely valued function.
- $b \neq 0$:
$$|f(z)| = e^{a\log|z|-b\mathrm{Arg}(z)} = e^{a\log|z|-b\arg(z)+2k\pi}$$
This indicate that $f(z)$ is an infinitely valued function.

Theorem 2.7 (Root of polynomials) *Abstract*
$$F(z) = \sqrt[n]{R(z)}$$
where
$$R(z) = \prod_{i=1}^{m}(z-a_i)^{n_i},\ n_i \in \mathbb{Z}$$

Then $F(z)$ has a single valued branch in Ω if and only if every simple closed curve C:
$$n \left| \sum_{a_j\, in\, C} n_j \right.$$
Furthermore, ∞ is a branch point of $F(z)$ if and only if
$$n \left| \sum_{j=1}^{m} n_j \right.$$

2.4.4 Triangular function

Definition 2.15 (Triangular function)

$$\cos z = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$
$$\sin z = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

2.5 Linear Fractional Transformation

Definition 2.16 (LFT) *A linear fractional transformation is shaped like*

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

Definition 2.17 *Four simple LFTs:*

1. ***Translation:***
$$f(z) = z + b,\ b \in \mathbb{C}$$
2. ***Rotation:***
$$f(z) = e^{i\theta} z,\ \theta \in \mathbb{R}$$
3. ***Scaling:***
$$f(z) = rz,\ r > 0$$
4. ***Inversion:***
$$f(z) = \frac{1}{z}$$

Theorem 2.8 (Decomposition) *All LFTs are recombined by four simple transforms.*

Theorem 2.9 *A LFT is a bijection in $\mathbb{C}$.*

Theorem 2.10 *A LFT is an identity, if it has 3 fixed points.*

Theorem 2.11 (Existence and uniqueness) *Given different z_1, z_2, z_3 and different w_1, w_2, w_3, there is a unique LFT s.t.*

$$f(z_i) = w_i,\ i = 1, 2, 3$$

Definition 2.18 (Cross ratio)

$$(z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \frac{z_1 - z_3}{z_1 - z_2}$$

Theorem 2.12 *A cross ratios is invariant under a LFT.*

Theorem 2.13 (Circle remaining) *A LFT map a circle into a circle.*

Theorem 2.14 z_1, z_2, z_3, z_4 *are concylic if and only if*

$$Im(z_1, z_2, z_3, z_4) = 0$$

Definition 2.19 (Left and right) *if* z_1, z_2, z_3 *lies on a circle in order, we naturally define their* ***left hand side*** *and* ***right hand side****.*

Theorem 2.15 z *is on the left hand side of* z_1, z_2, z_3 *if and only if*

$$Im(z, z_1, z_2, z_3) > 0$$

z *is on the right hand side of* z_1, z_2, z_3 *if and only if*

$$Im(z, z_1, z_2, z_3) < 0$$

A following corollary is that:

If z is on the left hand side of z_1, z_2, z_3, then $f(z)$ is on the left hand side of $f(z_1), f(z_2), f(z_3)$, where $f(z)$ is a LFT. The same is true of "right hand side".

Definition 2.20 (Symmetric points) *Given a point* z_1 *not on circle* $C : |z - a| = R$, z_2 *is the symmetric point of* z_1 *if*

$$|z_1 - a||z_2 - a| = R^2$$

In other words,

$$z_2 = a + \frac{R^2}{\bar{z} - \bar{a}}$$

That's to say, $re^{i\theta}$ and $\frac{R^2}{r}e^{i\theta}$ are symmetric points.

Theorem 2.16 (Determination of symmetric points) z_1 *and* z_2 *are symmetric about*

$$A|z|^2 + B\bar{z} + \bar{B}z + C = 0$$

if

$$Az_1z_2 + B\bar{z_1} + \bar{B}z_2 + C = 0$$

Theorem 2.17 *Abstract circles* C_1, C_2*, and*

$$f : C_1 \mapsto C_2$$

A LFT maps symmetric points about C_1 *into symmetric points about* C_2*.*

Transform of regions Functions we mentioned above transform some special regions into other special regions.

- $f(z) = \frac{az+b}{cz+d}$: straight line/circle $\longrightarrow$ straight line/circle.
- $f(z) = e^z$: zonal region $\longrightarrow$ angular region.
- $f(z) = \mathrm{Log}(z)$: angular region $\longrightarrow$ zonal region .
- $f(z) = z^n$: $\theta \longrightarrow n\theta$.
- $f(z) = z^{\frac{1}{n}}$: $\theta \longrightarrow \frac{1}{n}\theta$.
- $f(z) = \frac{1}{2}(z + \frac{1}{z})$: a recombination of z^2 and $\frac{1+z}{1-z}$.

Univalent domains of cosine function

$$\cos z_1 = \cos z_2 \Longleftrightarrow z_1 + z_2 \neq 2k\pi,\ z_1 - z_2 \neq 2k\pi$$

3 COMPLEX INTEGRAL

3.1 Cauchy Integral Formula

Definition 3.1 (Rectifiable curve) *Curve γ, rectifiable, if is length*

$$L(\gamma) = \sup_{||\pi||} \sum_k |\gamma(t_k) - \gamma(t_{k-1})| < \infty$$

Definition 3.2 (Complex integral) *Let $f(t) = x(t) + i\,y(t)$ be a complex-valued continuous function in $[a, b]$, we define:*

$$\int_a^b f(t)\ \mathrm{d}t = \int_a^b x(t)\ \mathrm{d}t + i\int_a^b y(t)\ \mathrm{d}t$$

Operation of complex integral Let $f(t) = x(t) + \mathrm{i}\,y(t)$ be a complex-valued continuous function in $[a, b], c \in \mathbb{C}$:

$$\mathrm{Re}(\int_a^b f(t)\ \mathrm{d}t) = \int_a^b x(t)\ \mathrm{d}t$$
$$\mathrm{Im}(\int_a^b f(t)\ \mathrm{d}t) = \int_a^b y(t)\ \mathrm{d}t$$
$$c\int_a^b f(t)\ \mathrm{d}t = \int_a^b cf(t)\ \mathrm{d}t$$

Absolute value inequality

$$|\int_a^b f(t)\ \mathrm{d}t| \leq \int_a^b |f(t)|\ \mathrm{d}t$$

Definition 3.3 (Curvilinear integral) *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be smooth, $f(z)$ be a continuous conplex-valued function, and we define:*

$$\int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t$$

As for a piecewise smooth curve, we similarly define:

$$\int_\gamma f(z)\ \mathrm{d}z = \sum_k \int_{t_k}^{t_{k-1}} f(\gamma(t))\gamma'(t)\ \mathrm{d}t$$

*As for a rectifiable curve, the definition returns to a **Riemann sum**:*

$$\int_\gamma f(z)\ \mathrm{d}z = \lim_{||\pi||\to 0} \sum_k f(z_k)|z_k - z_{k-1}|$$

Theorem 3.1 (Estimation lemma)

$$|\int_\gamma f(z)\ \mathrm{d}z| \leq ML(\gamma)$$

where

$$M = \sup_{z\in\gamma} |f(z)|$$

Respective integral Let $f = u + \mathrm{i}\,v, \gamma = x + \mathrm{i}\,y$:

$$\begin{aligned}
\int_\gamma f(z)\ \mathrm{d}z &= \int_a^b (u(\gamma) + \mathrm{i}\,v(\gamma))(x' + \mathrm{i}\,y')\ \mathrm{d}t \\
&= \int_a^b (ux' - vy' + \mathrm{i}(vx' + uy'))\ \mathrm{d}t \\
&= \int_a^b (u\ \mathrm{d}x - v\ \mathrm{d}y + \mathrm{i}(v\ \mathrm{d}x + u\ \mathrm{d}y)) \\
&= \int_a^b (u + \mathrm{i}\,v)(\ \mathrm{d}x + \mathrm{i}\ \mathrm{d}y))
\end{aligned}$$

Theorem 3.2 (Green's formula) *Let Ω be a simply connected region, $P, Q \in C^1(\Omega)$:*

$$\int_{\partial\Omega} P\ \mathrm{d}x + Q\ \mathrm{d}y = \int_\Omega (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\ \mathrm{d}x\ \mathrm{d}y$$

Theorem 3.3 (Cauchy integral theorem) *Abstract a simply connected region Ω, $f(z) \in H(\Omega)$. For each rectifiable closed curve $\gamma \subset \Omega$,*

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

Additionally, if $f(z)$ is continuous in $\bar{\Omega}$, we deduce

$$\int_{\partial\Omega} f(z)\ \mathrm{d}z = 0$$

Theorem 3.4 (Cauchy integral theorem in multiple connected regions) *Abstract a region D surrounded by $n+1$ simple closed curves $\gamma_0, \gamma_1, \cdots, \gamma_n$ and define $\partial D = \gamma_0^+ \cup \gamma_1^- \cup \cdots \cup \gamma_n^-$,*

$$f \in H(D) \cap C(\bar{D}) \Longrightarrow \int_{\partial D} f(z)\ \mathrm{d}z = \int_{\gamma_0} f(z)\ \mathrm{d}z - \sum_{k=1}^{n} \int_{\gamma_k} f(z)\ \mathrm{d}z = 0$$

3.2 Complex Fundamental Theorem of Calculus

Definition 3.4 (Primitive) *Given $f(z)$, if $F(z) \in H(\Omega)$ and $F'(z) = f(z)$, we call $F(z)$ the primitive of $f(z)$.*

Theorem 3.5 (Complex Newton-Leibniz formula) *Let $\gamma : [a,b] \longrightarrow \mathbb{C}$ be piecewise smooth. If $f(z)$ has a primitive $F(z)$,*

$$\int_\gamma f(z)\ \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

An immediate corollary is that:

Given a $f(z)$ defined above and a closed curve γ,

$$\int_\gamma f(z)\ \mathrm{d}z = 0 \tag{2}$$

Meanwhile, if $\exists$ a closed curve γ_0 s.t.

$$\int_\gamma f(z)\ \mathrm{d}z \neq 0$$

$f(z)$ has no primitive.

Derivative of constant function

$$f(z) \in H(\Omega),\ f'(z) = 0 \Longrightarrow f(z) = Const$$

Deriving a primitive Let Ω be a simply connected region, $z_0 \in \Omega, f(z) \in H(\Omega)$,

$$F(z) = \int_{z_0}^{z} f(w)\ \mathrm{d}w \in H(\Omega),\ F'(z) = f(z)$$

Existence of primitives in multiple connected region Let $f(z) \in H(\Omega) \cap C(\bar{\Omega})$, then $f(z)$ has a primitive in Ω if and only if

$$\int_{\gamma_k} f(z)\ \mathrm{d}z = 0,\ \forall k = 1, 2, \cdots, n$$

Single valued branch of a primitive Let $\Omega_j \subset \Omega$ are simply connected regions s.t.

$$\int_{\partial\Omega_j} f(z)\ \mathrm{d}z \neq 0$$

then all single valued branch of $F(z)$ are

$$F_0(z) + \sum_{j=1}^{n} n_j \int_{\partial\Omega_j} f(z)\ \mathrm{d}z,\ n_j \in \mathbb{Z}$$

3.3 Cauchy Integral Formula

Theorem 3.6 (Cauchy integral formula) *Let $f(z) \in H(\Omega) \cap C(\bar{\Omega}), \gamma$ be a simple closed curve:*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z}\ \mathrm{d}w$$

Namely, internal values are determined by boundary values.

Theorem 3.7 (Cauchy integral formula) *Let curve $\gamma \subset \Omega, \phi(z) \in C(\gamma), F(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(w)}{w-z}\ \mathrm{d}w$, we have:*

- *$F(z) \in H(\Omega \backslash \gamma)$.*
- *$F^{(n)}(z)$ exists, and*

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{\phi(w)}{(w - z)^{n+1}}\ \mathrm{d}w.$$

This theorem provides a method of integrating:

$$\int_{\gamma} \frac{f(w)}{(w - z)^m}\ \mathrm{d}w = \frac{2\pi \mathrm{i}}{(m - 1)!} f^{(m-1)}(z)$$

Theorem 3.8 (Cauchy's inequality) *For bounded function $f(z) \in H(B_R(a))$,*

$$|f^{(n)}(a)| \leq \frac{n!}{R^n} M$$

where M is a upper bound of $|f(z)|$ in $B_R(a)$, and n is an integer.

Definition 3.5 (Entire function) *A holomorphic function in the entire complex plane is called a* ***entire function****.*

Theorem 3.9 (Liouville theorem) *Bounded entire functions are constants.*

Theorem 3.10 (Fundamental theorem of algebra) *A polynomial function with complex coefficients has at least one complex root in complex field.*

Theorem 3.11 (Morera theorem) *Abstract $f(z) \in C(\Omega)$,*

$$\int_\gamma f(z)\ \mathrm{d}z = 0, \forall \gamma \subset \Omega \Longrightarrow f(z) \in H(\Omega)$$

Definition 3.6 (Boundless region surrounded by curves) *As for simple closed curves $\gamma_1, \gamma_2, \cdots, \gamma_n$, their joint exterior D is called the boundless region surrounded by $\gamma_1, \gamma_2, \cdots, \gamma_n$, and its boundary is $\partial D = \gamma_1^- \cup \gamma_2^- \cup \cdots \cup \gamma_n^-$.*

Theorem 3.12 (Cauchy theorem in boundless region) *Abstract boundless region D surrounded by curve γ, $f(z) \in H(D) \cap C(\bar{D})$ s.t.*

$$\lim_{z\to\infty} z^2 f(z) = a \in \mathbb{C}$$

we have

$$\int_\gamma f(z)\ \mathrm{d}z = 0$$

Theorem 3.13 (Cauchy's formula in boundless region) *Abstract boundless region Ω surrounded by curve γ and $f(z) \in H(\Omega) \cap C(\bar{\Omega}), f(\infty) \in \mathbb{C}$, then*

$$f(z) = f(\infty) + \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\ \mathrm{d}w$$
$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\ \mathrm{d}w$$

Theorem 3.14 (A variant of Cauchy's inequality) *Abstract $f(z) \in H(B_R(a))$, we have*

$$|f'(a)| \leq \frac{2M}{R}$$

where $|Re(f(z))| \leq M$.

4 COMPLEX SERIES

4.1 Convergence of Complex Series

Definition 4.1 (Convergence of complex series) *Abstract complex series $\{z_n\}$, let*

$$S_n = \sum_{k=1}^{n} z_k$$

The series $\sum_{n=1}^{\infty} z_n$ is convergent if

$$S_n \to S,\ n \to \infty$$

Equivalently, $\sum_{n=1}^{\infty} z_n$ is convergent if and only if the following two real series

$$\sum_{n=1}^{\infty} \mathrm{Re}(z_n) \qquad \sum_{n=1}^{\infty} \mathrm{Im}(z_n)$$

are convergent.

Theorem 4.1 (Cauchy's criterion of convergence) *$\sum_{n=1}^{\infty} z_n$ is convergent if and only if for every $\varepsilon > 0$, there exists a $N > 0$ such that*

$$n_2 > n_1 > N \Longrightarrow |z_{n_1+1} + \cdots + z_{n_2}| < \varepsilon$$

Definition 4.2 (Convergence of complex series of functions) *Abstract complex series of functions $\{f_n(z)\}$, the series $\sum_{n=1}^{\infty} f_n(z)$ is convergent if for all z in E, $\sum_{n=1}^{\infty} f_n(z)$ is convergent.*

Additionally, The **sum function** of the series is define as

$$S(z) = \sum_{n=1}^{\infty} f_n(z)$$

Definition 4.3 (Uniform convergence) *$S_n(z)$ uniformly converge to $S(z)$ if for every $\varepsilon > 0$, there exists a $N > 0$ such that for all z in E,*

$$n > N \Longrightarrow |S_n(z) - S(z)| < \varepsilon$$

While **Cauchy's criterion of convergence** and **Weierstrass test** holds for uniform convergence.

Definition 4.4 (Internally closed uniform convergence) *$\sum_{n=1}^{\infty} f_n(z)$ is Internally closed uniformly convergent if it is uniformly convergent in each compact set in E.*

4.2 Power Series

Definition 4.5 (Convergence radius) *For power series $\sum_{n=0}^{\infty} a_n z^n$, its convergence radius is*

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}}$$

The convergence radius shows that

$$|z| < R \Longrightarrow \sum_{n=0}^{\infty} a_n z^n < +\infty$$

$$|z| > R \Longrightarrow \sum_{n=0}^{\infty} a_n z^n = +\infty$$

Theorem 4.2 (Holomorphism of power series) *Abstract a function in form of power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

whose convergence radius is R. Then

1. $f(z) \in H(B_R(0))$
2. $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$
3. $f'(z)$ *is convergent and holomorphic in* $B_R(0)$

Additionally, we deduce that power series is infinitely derivable in its convergence domain.

Definition 4.6 (Analyticity) $f(z)$ *is* ***analytic*** *at* z_0 *if it can expand into power series at* z_0*, and it is analytic in D if it is analytic at every point in D.*

Theorem 4.3 (Abel theorem) *Abstract*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

whose convergence radius is $R = 1$*. If it converge to S at* $z = 1$*,* $f(z)$ *is uniformly convergent in* $\bar{A}$ *and*

$$\lim_{\substack{|z|<1 \\ z \to 1}} f(z) = S$$

where

$$A = \{z \,|\, \pi - \theta_0 < \arg(z-1) < \pi + \theta_0\} \cap B_1(0), \; \theta_0 \in (0, \frac{\pi}{2})$$

is an angular region.

Theorem 4.4 *If series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

is convergent at $z_0 = x_0 + iy_0$*, then*

1. *It converges in* $\{z \,|\, Re(z) > x_0\}$
2. *It uniformly converges in* $\bar{A}$*, where* $A = \{z \,|\, \theta_0 < \arg(z - z_0) < \theta_0\}$
3. *There exists a convergence line* $Re(z) = C$*, it converge when* $Re(z) > C$

Definition 4.7 (Riemann's zeta function) *The convergence line of series*

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

is $Re(z) = 1$*. It can be extended to* $\mathbb{C} \backslash \{1\}$*, called Riemann's zeta function.*

Theorem 4.5 (Termwise integration) *If* $f_n(z)$ *is continuous in curve* γ *and*

$$\sum_{n=1}^{\infty} f_n(z) \rightrightarrows f(z)$$

then we have

$$\int_{\gamma} f(z) \, \mathrm{d}z = \sum_{n=1}^{\infty} \int_{\gamma} f_n(z) \, \mathrm{d}z$$

Theorem 4.6 (Uniform convergence of higher order derivatives) *For* $f_n(z) \in H(\Omega)$*, if*

$$\sum_{n=1}^{\infty} f_n(z)$$

internally closed uniformly converge to $f(z)$

1. $f(z) \in H(z)$
2. $\sum_{n=1}^{\infty} f_n^{(m)}(z)$ *internally closed uniformly converge to* $f^{(m)}(z)$

Theorem 4.7 (Holomorphism of multivariate function) *If*

$$F(z, s) : \Omega \times [0, 1] \longrightarrow \mathbb{C}$$

satisfying

1. $\forall\, s \in [0, 1], F(z, s)$ *is a holomorphic function of* z
2. $F(z, s) \in C(\Omega \times [0, 1])$

the function

$$f(z) = \int_0^1 F(z, s) \, \mathrm{d}s$$

is holomorphic in Ω*.*

4.3 Taylor Expansion of Holomorphic Functions

Theorem 4.8 (Taylor expansion) *Let $f(z) \in H(\Omega), \bar{B}_r(z_0) \subset \Omega$, then $f(z)$ can uniquely expand into a power series*

$$f(z) = \sum_{n=0}^{n} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{n!} f^{(n)}(z_0), \; z = B_r(z_0)$$

This theorem implies that holomorphism is equivalent to analyticity.

Theorem 4.9 (Multiplicity of zero) *For $f(z) \in H(z_0), f(z_0) = 0$, we have following results:*

$$f^{(n)}(z_0) = 0, \forall\, n \Longrightarrow a_0 = a_1 = \cdots = 0 \Longrightarrow f(z) = 0, \forall\, z \in B_\delta(z_0)$$

$$f^{(n)}(z_0) = 0, \forall\, n \leq m, f^{(m)}(z_0) \neq 0 \Longrightarrow f(z) = a_m (z - z_0)^m (1 + \frac{a_{m+1}}{a_m} z + \cdots)$$

In the second situation, z_0 is a zero of multiplicity m.

Theorem 4.10 (Isolatism of zeros) *The set of zeros of a holomorphic, i.e. for $f(z) \in H(\Omega)$*

$$\exists\, \{z_n\}_{n=1}^{\infty} \subset \Omega, z_n \to z_0 \in \Omega, f(z_n) = 0, \forall\, n \Longrightarrow f(z) = 0, \forall z \in \Omega$$

Theorem 4.11 (Uniqueness theorem) *For $f_1(z), f_2(z) \in H(\Omega)$, if*

$$\exists\, \{z_n\} \subset \Omega, z_n \to z_0, z_n \neq z_0, f_1(z_n) = f_2(z_n)$$

then $f_1(z) = f_2(z)$ in Ω.

4.4 Laurent Expansion

4.4.1 Laurent series

Definition 4.8 (Laurent series)

$$\sum_{n=-\infty}^{+\infty} a_n (z - z_0)^n = \sum_{n=0}^{+\infty} a_n (z - z_0)^n + \sum_{n=-\infty}^{-1} a_n (z - z_0)^n \tag{3}$$

The part similar to Taylor series is its **holomorphic part**, while the other part is its **main part**. The series is convergent if and only if the two parts above are convergent.

Convergence domain Suppose the convergence radius of the holomorphic part is R, and let $w = \frac{1}{z-z_0}$, then the main part

$$\sum_{n=-\infty}^{-1} a_n(z-z_0)^n = \sum_{n=1}^{+\infty} a_n w^n$$

Suppose the convergence radius of the series of w is ρ, then the convergence domain of series (3) is an annulus

$$\{z \,|\, \frac{1}{\rho} < |z-z_0| < R\}$$

Theorem 4.12 (Laurent expansion) *Abstract a holomorphic function $f(z)$ in annulus $\{z \,|\, r < |z-z_0| < R\}$, it can expand into a unique Laurent series in form of (3), where*

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(w)}{(w-z_0)^{n+1}} \, \mathrm{d}w$$

and $r < \rho < R$.

4.4.2 Isolated singularity

Classification If $f(z) \in H(\check{B}_r(z_0))$ has no definition at point z_0, we call z_0 an **isolated singularity** of $f(z)$. Moreover, we can classify them into three type by the Laurent expansion of $f(z)$:

1. **Removable singularity:** No negative power term
2. **Pole:** Finite negative power terms
3. **Essential singularity:** Infinite negative power terms

Removable singularity Following propositions are equivalent:

1. z_0 is a removable singularity of $f(z)$
2. $f(z)$ is bounded in a deleted neighborhood of z_0
3. The Laurent expansion of $f(z)$ at z_0 has no negative power term
4. The limit
$$\lim_{z \to z_0} f(z)$$
exists and is finite

Pole Following propositions are equivalent:

1. z_0 is a pole of $f(z)$
2. $\lim\limits_{z\to z_0} f(z) = \infty$
3. $\exists\, m > 0$, s.t. $g(z) = (z - z_0)^m f(z)$ is a holomorphic function in $B_\delta(z_0)$ and has no zero in $B_\delta(z_0)$
4. The Laurent expansion of $f(z)$ at z_0 has finite negative power terms
5. $\exists\, m > 0$, s.t. the limit
$$\lim_{z\to z_0} (z - z_0)^m f(z)$$
exists and isn't 0.
6. $\exists\, m > 0$, s.t. z_0 is an mth-order zero of $g(z) = \frac{1}{f(z)}$

Here m is called the order of pole z_0.

Essential singularity Following propositions are equivalent:

1. z_0 is an essential singularity of $f(z)$
2. $\lim_{z\to z_0}$ doesn't exist
3. The Laurent expansion of $f(z)$ at z_0 has infinite negative power terms
4. $\forall\, A \in \bar{\mathbb{C}}, \exists\,\{z_n\} \subset \check{B}_\delta(z_0)$, s.t.
$$\lim_{n\to\infty} f(z_n) = A$$

Infinite isolated singularity If $f(z)$ is a holomorphic function in $\{z \,|\, |z| > R\}$, ∞ is an isolated singularity of $f(z)$. And if $z = \infty$ is an isolated singularity of $f(z)$, the type of $z = \infty$ of $f(z)$ is identical to that of $z = 0$ of $f(\frac{1}{z})$. For example,

- ∞ isn't an isolated singularity of
$$f(z) = \frac{1}{\sin z}$$
- ∞ is an nth-order pole of
$$f(z) = \sum_{k=0}^{n} a_k z^k$$
where $a_n \neq 0$

4.5 Meromorphic Function

Definition 4.9 (Entire function) *A holomorphic function in* $\mathbb{C}$ *is an entire function.*

Infinite isolated singularity of entire functions $z = \infty$ is an isolated singularity of an entire function $f(z)$, and

- $z = \infty$ is a removable singularity of $f(z) \Longleftrightarrow$ if $f(z) = Const$
- $z = \infty$ is a pole of $f(z) \Longleftrightarrow$ if $f(z)$ is a polynomial
- $z = \infty$ is an essential singularity of $f(z)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ \overline{\lim_{n\to\infty}} |a_n| > 0$$

Definition 4.10 (Meromorphism) *$f(z)$ is meromorphic in region $D \in \mathbb{C}$, if $f(z)$ is holomorphic in every point in D except its poles.*

Theorem 4.13 (Rational function) *Every meromorphic function can be presented in form of rational function, i.e.*

$$f(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are two polynomials.

Theorem 4.14 *A meromorphic bijection in* $\mathbb{C}$ *is an LFT.*

Theorem 4.15 *If the inverse function of an entire function $f(z)$ exists and is entire, we have*

$$f(z) = az + b$$

5 ESTIMATION OF FUNCTIONS

5.1 Maximum Modulus Principle

Theorem 5.1 (Maximum modulus principle) *If $f(z) \in H(D)$ is nonconstant, $|f(z)|$ doesn't reach it maximum in D.*

Another expression of this theorem is

Theorem 5.2 *For $f(z) \in H(D) \cap C(\bar{D})$, $\forall\, z \in D$,*

$$|f(z)| \le \max_{w\in\partial D} |f(w)|$$

Theorem 5.3 (Extremum principle of harmonic functions) *If $u(x, y)$ is a harmonic and nonconstant function in region D, $u(x, y)$ dosen't reach its maximum and minimum in D.*

Abstracting $g(z) = \frac{1}{f(z)}$, we can deduce a useful proposition:

Theorem 5.4 (Minimum principle) *If $f(z) \in H(D)$ is nonconstant and $f(z) \neq 0, \forall\, z \in D$, then $f(z)$ doesn't reach it minimum in D.*

5.2 Schwartz's Lemma

Theorem 5.5 (Schwartz's lemma) *Abstract holomorphic function in unit disc $\mathbb{D} = \{z \,|\, |z| < 1\}$:*

$$f : \mathbb{D} \longrightarrow \mathbb{D}$$

and $f(0) = 0$, we can deduce following conclusions:

1. $|f(z)| \leq |z|, \forall\, z \in \mathbb{D}$
2. *If $|f(z_0)| = |z_0|, z_0 \neq 0$, then $f(z)$ is a rotation, i.e $\exists\, \theta_0 \in \mathbb{R}$, s.t*

$$f(z) = e^{i\theta_0} z$$

3. *$|f'(0)| \leq 1$, and the equility holds if and only if $f(z)$ is a rotation*

Definition 5.1 (Conformal equivalence) *Abstract holomotphic bijection*

$$f : U \longrightarrow V$$

It is a conformal mapping, then U and V are conformal equivalent or biholomorphic equivalent.

Definition 5.2 (Holomorphic automorphism) *A conformal function*

$$f : U \longrightarrow U$$

is called a holomorphic automorphism of U.

All holomorphic automorphism of U compose a group $Aut(U)$.

Theorem 5.6 (Particular holomorphic automorphisms)

$$Aut(\mathbb{C}) = \{az + b \,|\, a \neq 0, b \in \mathbb{C}\}$$
$$Aut(\bar{\mathbb{C}}) = \{\frac{az + b}{cz + d} \,|\, ad - bc \neq 0\}$$
$$Aut(\mathbb{D}) = \{e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z} \,|\, \theta \in \mathbb{R}, |\alpha| < 1\}$$

5.3 Argument Principle

Definition 5.3 (Argument Principle) *Let $N(f,\gamma)$ be the total orders of the zeros of $f(z)$ in simple closed curve γ, $P(f,\gamma)$ be that of poles, then*

$$\begin{aligned} N(f,\gamma)-P(f,\gamma) &= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\ \mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}f(z)}{f(z)} \\ &= \frac{1}{2\pi i}\int_\sigma \frac{\mathrm{d}w}{w} = \frac{1}{2\pi}\Delta_\sigma Arg(w) = \frac{1}{2\pi}\Delta_\gamma Arg(z) \end{aligned}$$

Theorem 5.7 (Rouché's theorem) *If $f(z),g(z)\in H(D),\gamma\subset D$ satisfy*

$$|g(z)|<|f(z)|,\ \forall\, z\in\gamma$$

the number of zeros of $f(z)$ and $f(z)\neq g(z)$ is the same in the region surrounded by γ.

Definition 5.4 (Open mapping) *A open mapping maps an open set into an open set, i.e.*

$$f(B_\delta(z_0))\supset B_\rho(w_0)$$

Specifically, holomorphic functions are open mappings.

Theorem 5.8 *Abstract a nonconstant function $f(z)\in H(\Omega), f(0)=0$. For a sufficiently small $\rho>0, \exists>\ \delta>0$, when $0<|w_0|<\delta$, $f(z)-w_0$ has a zero in $B_\rho(0)$. Additionally, if 0 is an mth-order zero, the result above is improved into $f(z)-w_0$ has m zeros in $B_\rho(0)$.*

Theorem 5.9 *For nonconstant $f(z)\in H(\Omega), f(z_0)=w_0$ and sufficiently small $\rho>0$, $\exists\,\delta>0$, s.t.*

$$B_\delta(w_0)\subset B_\rho(z_0)$$

Theorem 5.10 *A nonconstant holomorphic function maps regions into regions.*

Theorem 5.11 (Differential property of univalent functions) *For a univalent function $f(z)$ in Ω,*

$$f'(z)\neq 0, \forall\, z\in\Omega$$

Inversely, if $f(z_0)\neq 0, \exists\,\varepsilon_0>0$, s.t. $f(z)$ is univalent in $B_{\varepsilon_0}(z_0)$.

Theorem 5.12 (Inverse function of univalent functions) *The inverse function of a univalent holomorphic function is a univalent holomorphic function.*

5.4 Residue Theorem

Definition 5.5 (Residue) *For $f\in H(\check{B}_r(a))$, we define its residue at a:*

$$Res(f,a)=\frac{1}{2\pi i}\int_{|z-a|=\rho} f(z)\ \mathrm{d}z$$

where $\rho\in(0,r)$.

In particular, if the Laurent expansion of f at a is

$$f(z) = \sum_{n=-\infty}^{+\infty} = c_n(z-a)^n$$

then $\text{Res}(f, a) = c_{-1}$.

Residue around singularity The value of a residue around a sigularity a relies on the type of the singularity.

1. Removable singularity: $\text{Res}(f, a) = 0$
2. First-order Pole: $\text{Res}(f, a) = c_{-1}$
3. mth-order singularity: $\text{Res}(f, a) = \frac{((z-a)^m f(z))^{(m-1)}}{(m-1)!}|_{z=a}$

Theorem 5.13 (Residue theorem) *For $f \in H(D\backslash\{z_1, \cdots, z_n\}) \cap C(\bar{D}\backslash\{z_1, \cdots, z_n\})$, we have*

$$\int_{\partial D} f(z)\ \mathrm{d}z = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

Definition 5.6 (Infinite residue) *For $f \in H(\mathbb{C}\backslash B_R(0))$, we define*

$$Res(f, \infty) = -2\pi i \int_{|z|=\rho} f(z)\ \mathrm{d}z$$

where $\rho > R$.

Theorem 5.14 (Infinity residue theorem) *For $f \in H(D\backslash\{z_1, \cdots, z_n\})$, the sum of residues of f around all its isolated singularity is 0, i.e.*

$$Res(f, \infty) + \sum_{k=1}^{n} Res(f, z_k) = 0$$

6 HOLOMORPHIC EXTENSION

6.1 Schwartz's Principle of Symmetry

Definition 6.1 (Holomorphic extension) *For region D and $f \in H(D)$, if there exists a region $G \supset D$ and $F \in H(G)$, s.t.*

$$F(z)|_D = f(z)$$

then F is called the holomorphic extension of f in G.

Theorem 6.1 (Painlevé's theorem) *If region Ω is divided into two regions Ω_1, Ω_2 by curve γ, then*

$$f \in H(\Omega_1 \cup \Omega_2) \cap C(\Omega) \Longrightarrow f \in H(\Omega)$$

This theorem implies a critical corollary,

$$\left.\begin{aligned} f_1(z) \in H(\Omega_1) \cap C(\Omega_1 \cap \gamma) \\ f_1|_\gamma = f_2|_\gamma \end{aligned}\right\} \Longrightarrow F(z) \in H(\Omega)$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \gamma$, and

$$F(z) = \begin{cases} f_1(z), & z \in \Omega_1 \cup \gamma \\ f_2(z), & z \in \Omega_2 \end{cases} \tag{4}$$

Theorem 6.2 (Schwarz's symmetry theorem) *Let D be symmetric about real axis, f satisfies*

1. *f is holomorphic in $D \cap \{z \in \mathbb{C} | Im(z) > 0\}$*
2. *f is continuous in $D \cap \{z \in \mathbb{C} | Im(z) \geq 0\}$*
3. *f is real-valued in $D \cap \{z \in \mathbb{C} | Im(z) = 0\}$*

then

$$F(z) = \begin{cases} f(z), & z \in D \cap \{z \in \mathbb{C} | Im(z) \geq 0\} \\ \overline{f(\bar{z})}, & z \in D \cap \{z \in \mathbb{C} | Im(z) < 0\} \end{cases} \tag{5}$$

is the holomorphic extension of f in D.

We can promote the "axis of symmetry" of the theorem from real axis into arbitrary general circles.

Theorem 6.3 *Suppose*

1. *Ω and Ω' are symmetric about circle $S = \{z||z| = r\}$*
2. *$f(z)$ is holomorphic in Ω*
3. *$f(S)$ is an arc Γ*
4. *The center of Γ, $b \notin f(\Omega)$*

then $f(z)$ can be holomorphically extended into $\Omega \cup S \cup \Omega'$.

6.2 Holomorphic Extension of Power Series

Definition 6.2 (Regular point and singular point) *Abstract a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

with a convergence radius R. It is holomorphic in $D = \{z||z| < R\}$

z_0 is a regular point of $f(z)$, if for $z_0 \in \partial D$,there exists a neighborhood $B_\delta(z_0)$ and holomorphic function $g(z)$ in it, s.t

$$f(z) = g(z),\ \forall z \in D \cap B_\delta(z_0)$$

z_0 is a singular point of $f(z)$, if $z_0 \in \partial D$ is not a regular point.

If $f \in H(D_1), g \in H(D_2), D_1 \cap D_2 \neq \varnothing$, and $f_{D_1 \cap D_2} = g_{D_1 \cap D_2}$, we denote

$$(f, D_1) \sim (g, D_2)$$

Theorem 6.4 (Existence of singular point) *There is at least* 1 *singular point on the convergence circle of a power series.*

Theorem 6.5 (Determine type of points) *To identify the regular and singular points of power series of $f(z)$, we have following theorems.*

1. *if $\lim_{z \to z_0} |f(z)| = +\infty$, z_0 is a singular point*
2. *$f(z)$ and $f'(z)$ have identical regular and singular points on $|z| = R$*
3. *There is no necessary connection between regular/singular points and convergent/diverging points*

There are 4 *typical examples for item* 3.

	convergent point	*diverging point*
regular point	$z = 1$ *of* $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{z^n}{n}$	$z = -1$ *of* $\sum_{n=0}^{\infty} z^n$
singular point	$z = 1$ *of* $\sum_{n=0}^{\infty} \frac{z^n}{n(n-1)}$	$z = 1$ *of* $\sum_{n=0}^{\infty} z^n$

Theorem 6.6 *Under the condition of*

$$\lim_{z \to z_0} |f(z)| \neq \infty$$

we pick a point z_0' on the segment between O and z_0 and suppose that ρ is the convergence radius of

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(}n)(z_0')}{n!} (z - z_0')^n$$

Apparently, $\rho \geq R - |z_0'|$. Moreover, z_0 is a regular point if $\rho > R - |z_0'|$, z_0 is a singular point if $\rho = R - |z_0'|$.

7 RIEMANN CONFORMAL MAPPING

7.1 Regular Family

Theorem 7.1 (Hurwitz's theorem) *If*

1. *$f_n(z) \in H(D)$ internally closed uniformly converge to $f(z)$, $f(z)$ is nonzero*
2. *Closed curve γ doesn't pass zeros of $f(z)$*

there exists N, s.t. $f_n(z)$ has as many zeros as $f(z)$ in γ when $n > N$.

Theorem 7.2 *If*

1. $f_n(z) \in H(D)$ *is univalent*

2. $f_n(z)$ *internally closed uniformly converge to* $f(z)$, $f(z)$ *is nonconstant*

then $f(z)$ *is univalent and holomorphic in* D.

Definition 7.1 (Internally closed uniform boundedness) *A family of functions* $\mathcal{F}$ *in* Ω, *is internally closed uniformly bounded, if* $\forall\, K \subset \Omega$, $\exists\, M(K) > 0$, *s.t.* $|f(z)| \leq M(K), \forall\, f \in \mathcal{F}$, *where* K *is compact.*

Definition 7.2 (Internally closed equicontinuity) *A family of functions* $\mathcal{F}$ *in* Ω, *is equicontinuous, if* $\forall\, K \subset \Omega, \forall\, \varepsilon > 0$, $\exists\, \delta(\varepsilon, K) > 0$, *s.t.* $|f(z_1) - f(z_2)| < \varepsilon, \forall\, f \in \mathcal{F}$ *as long as* $|z_1 - z_2| < \delta$ *where* K *is compact.*

Theorem 7.3 (Arzela-Ascoli lemma) *For compact set* $K \subset \mathbb{C}$, *if* $\{f_n\}$ *is uniformly bounded and equicontinuous in* K, *there exists a subsequence of* $\{f_n\}$ *which uniformly converges to continuous function* f.

Theorem 7.4 (Montel's theorem) *A family of internally closed uniformly bounded holomorphic functions* $\mathcal{F}$ *in* D *has a subsequence which internally closed uniformly converges in* D.

Definition 7.3 (Regular family) *A family of functions is call a regular family if an arbitrary sequence of function in the family has an internally closed uniformly convergent subsequence.*

7.2 Riemann Mapping Theorem

Theorem 7.5 (Riemann mapping theorem) *A simply connected region* $\Omega \subset \mathbb{C}$ *is holomorphically isomorphic to* $\mathbb{D}$

Theorem 7.6 (Reinforced Riemann mapping theorem) *For a simply connected region* $\Omega \subset \mathbb{C}$, *there exists a unique holomorphic bijection* $F : \Omega \longrightarrow \mathbb{D}$, *s.t* $F(z_0) = 0, F'(z_0) > 0$

For a simply connected U and $f \in H(U)$, $f(U)$ is not necessarily simply connected. A counter-example is

$$f : \mathbb{H} \longrightarrow \mathbb{D} \backslash \{0\}, z \mapsto e^{2\pi \mathrm{i}\, z} \tag{6}$$

7.3 Boundary Correspondence

Theorem 7.7 (Theorem of boundary correspondence) *Abstract conformal function* $F : \mathbb{D} \longrightarrow P$, P *is an open polygon, we have*

1. F *can be continuously extended into a bijection from* $\overline{\mathbb{D}}$ *to* $\overline{\mathbb{P}}$

2. F *is a bijection from* $\partial\mathbb{D}$ *to* ∂P

Two counter-example of last theorem are

1. For $\Omega = ((0,1)\times(0,1))\backslash(\bigcup_{n=2}^{\infty}\frac{1}{n}\times\left(0,\frac{1}{2}\right])$, there is no continuous curve connecting $z\in\Omega$ and $(0,\frac{1}{4})$
2. For $\mathbb{D}\backslash\{x|0\le x<1\}$ and different n, there is no continuous curve connecting $z_n = (\frac{1}{2},(-1)^n\frac{1}{n})$

Theorem 7.8 *Abstract a simple closed curve $\gamma\subset D$, the region surrounded by γ is D_1. If $f\in H(D)$ map γ into a simple closed curve Γ univalently, then f is univalent in D_1, and maps D_1 into the region surrounded by Γ, positive direction of γ into that of Γ.*

7.4 Conformal Mapping of Polygons

Theorem 7.9 *$f(z)=z^{\alpha}, 0<\alpha<2$ maps $\mathbb{H}$ into an angular region and can be continuously extended to the boundary.*

Theorem 7.10 *A single valued branch of function $f(z)=\int_0^z\frac{\mathrm{d}\xi}{\sqrt{1-\xi^2}}$ maps $\mathbb{H}$ into $\{z|\frac{\pi}{2}<Re(z)<\frac{\pi}{2}, Im(z)>0\}$*

Theorem 7.11 *Let P be a polygon. F is a conformal function from $\mathbb{H}$ to P if and only if F has the form*

$$F(z)=c_1\int_0^z\frac{\mathrm{d}\xi}{(\xi-A_1)^{\beta_1}\cdots(\xi-A_n)^{\beta_n}}+c_2$$

8 FOURIER TRANSFORM

9 ENTIRE FUNCTIONS

10 GAMMA AND ZETA FUNCTION