Harmonic Analysis Homeworks

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1 Homework 1

1.1 Problem 1.1

If $f \in L^\infty$ and $\| \tau_y f - f \|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function ($\tau_y f(x) = f(x-y)$).

Proof. First of all, we consider the average integral

$$A_r f(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) \, \mathrm{d}y$$

By Lebesgue differentiation theorem, we have

$$F(x) = \lim_{r \to 0} A_r f(x) = f(x), \text{ a.e.}$$

In that case, we only need to show that $F(x)$ is continuous.

Let $F_n(x) = A_{\frac{1}{n}} f(x)$, and we are going to show $F_n(x)$ converges to $F(x)$ in L^∞ as $n \to \infty$. To simplify the notations, we denote $B_n = B_{\frac{1}{n}}(x)$ for fixed x, thus

$$\begin{aligned}
\| F_{n+p}(x) - F_n(x) \|_\infty &= \operatorname{ess\,sup} \left| \frac{1}{|B_{n+p}|} \int_{B_{n+p}} f(y) \, \mathrm{d}y - \frac{1}{|B_n|} \int_{B_n} f(y) \, \mathrm{d}y \right| \\
&\leq \operatorname{ess\,sup} \left| \frac{1}{|B_{n+p}|} \int_{B_{n+p}} f(y) \, \mathrm{d}y - f(x) \right| + \operatorname{ess\,sup} \left| \frac{1}{|B_n|} \int_{B_n} f(y) \, \mathrm{d}y - f(x) \right| \\
&\leq \operatorname{ess\,sup} \frac{1}{|B_{n+p}|} \int_{B_{n+p}} |f(y) - f(x)| \, \mathrm{d}y + \operatorname{ess\,sup} \frac{1}{|B_n|} \int_{B_n} |f(y) - f(x)| \, \mathrm{d}y \\
&\leq \| f(y) - f(x) \|_{L^\infty(B_{n+p})} + \| f(y) - f(x) \|_{L^\infty(B_n)} \\
&\leq 2 \| \tau_{x-y} f(x) - f(x) \|_{L^\infty(B_n)} \\
&= 2 \| \tau_y f(x) - f(x) \|_{L^\infty(B_{\frac{1}{n}}(0))} \\
&\to 0, \ n \to \infty
\end{aligned}$$

where p is a positive integer.

Note that this result is independent of the choice x, so $\{F_n(x)\}$ is a uniform Cauchy sequence whose limit is $F(x)$.

Additionally, we have

$$\begin{aligned}
|F_n(x+h) - F_n(x)| &= \left| \frac{1}{|B_{\frac{1}{n}}(x)|} \int_{B_{\frac{1}{n}}(x)} f(x+h) \, \mathrm{d}y - \frac{1}{|B_{\frac{1}{n}}(x)|} \int_{B_{\frac{1}{n}}(x)} f(x) \, \mathrm{d}y \right| \\
&\leq \frac{1}{|B_{\frac{1}{n}}(x)|} \int_{B_{\frac{1}{n}}(x)} |f(x+h) - f(x)| \, \mathrm{d}y \\
&\leq \| f(x+h) - f(x) \|_\infty \\
&\to 0, \ h \to 0
\end{aligned}$$

for $n > 0$, i.e. $F_n(x)$ is uniformly continuous.

$\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ and $N \in \mathbb{N}$, such that

$$|F_n(x) - F(x)| < \frac{\varepsilon}{3} \qquad |F_n(x_1) - F_n(x_2)| < \frac{\varepsilon}{3}$$

as long as $n > N, |x_1 - x_2| < \delta$. Since δ does not rely on the choice of x_1, x_2, we obtain

$$|F(x_1) - F(x_2)| \leq |F_n(x_1) - F(x_1)| + |F_n(x_1) - F_n(x_2)| + |F_n(x_2) - F(x_2)| < \varepsilon$$

which implies the uniform continuity of $F(x)$. ☐

1.2 Problem 1.2

(a) Prove that for all $0 < \varepsilon < t < +\infty$, we have

$$\left| \int_{\varepsilon}^{t} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| \leq 4$$

Proof. By the linearity of integral and triangular inequality, we only need to show that

$$|F(t)| = \left| \int_{\frac{\pi}{2}}^{t} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| \leq 2, \ \forall \, t > 0$$

with $F(\frac{\pi}{2}) = 0$, and

$$F'(t) = \frac{\sin t}{t}$$

Therefore, $|F|$ reaches its maximum at $k\pi$. Additionally, in $((k-1)\pi, k\pi)$, F increases for odd k, and decreases for even k.

By the uniform convergence of Taylor series, we have

$$|F(0)| = \left| \int_{0}^{\frac{\pi}{2}} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| = \left| \int_{0}^{\frac{\pi}{2}} \left(1 - \frac{\xi^2}{6} + \frac{\eta}{120} \right) \mathrm{d}\xi \right| \leq \left| \frac{\pi}{2} - \frac{\pi^3}{144} + \frac{\pi^5}{3840} \right| \leq \pi - \frac{\pi^3}{18} + \frac{\pi^5}{120} < \frac{3}{2}$$

and

$$|F(\pi)| = \left| \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| \leq \left| \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\pi - \xi} \, \mathrm{d}\xi \right| = \left| \int_{0}^{\frac{\pi}{2}} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| < \frac{3}{2}$$

For $k \geq 2$, apparently

$$\left| \int_{(k-1)\pi}^{k\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right| > \left| \int_{k\pi}^{(k+1)\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right|, \ \forall \, k$$

and the two terms differ in sign. Therefore,

$$\begin{aligned}
F((2k+1)\pi) &= \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \sum_{j=2}^{2k+1} \int_{(j-1)\pi}^{j\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \sum_{j=1}^{k} \left(\int_{(2j-1)\pi}^{2j\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \int_{2j\pi}^{(2j+1)\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right) \\
&< \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi < \frac{3}{2}
\end{aligned}$$

and

$$\begin{aligned}
F(2k\pi) &= \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \sum_{j=2}^{2k} \int_{(j-1)\pi}^{j\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \\
&= \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \int_{\pi}^{2\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \sum_{j=1}^{k} \left(\int_{(2j-2)\pi}^{(2j-1)\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi + \int_{(2j-1)\pi}^{2j\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \right) \\
&\geq \int_{\pi}^{2\pi} \frac{\sin \xi}{\xi} \, \mathrm{d}\xi \geq - \int_{\pi}^{2\pi} \frac{1}{\xi} = -\log 2 > -1
\end{aligned}$$

Ultimately, we have obtained the conclusion

$$|F(t)| \leq \sup_{k \in \mathbb{N}} |F(k\pi)| \leq 2$$

□

(b) If f is an odd L^1 function on the line, conclude that for all $t > \varepsilon > 0$ we have

$$\left| \int_\varepsilon^t \frac{\hat{f}(\xi)}{\xi} \,\mathrm{d}\xi \right| \leq 4 \parallel f \parallel_1$$

Proof. By Fubini theorem,

$$\begin{aligned}
\int_\varepsilon^t \frac{\hat{f}(\xi)}{\xi} \,\mathrm{d}\xi &= \int_\varepsilon^t \frac{1}{\xi} \left(\int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} \,\mathrm{d}x \right) \mathrm{d}\xi \\
&= \int_{-\infty}^{+\infty} f(x) \left(\int_\varepsilon^t \frac{\cos 2\pi x\xi - i \sin 2\pi x\xi}{\xi} \,\mathrm{d}\xi \right) \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} f(x) \left(\int_\varepsilon^t \frac{\cos 2\pi x\xi}{\xi} \,\mathrm{d}\xi \right) \mathrm{d}x - i \int_{-\infty}^{+\infty} f(x) \left(\int_\varepsilon^t \frac{\sin 2\pi x\xi}{\xi} \,\mathrm{d}\xi \right) \mathrm{d}x
\end{aligned}$$

It is not hard to see that the definite integral of $\frac{\cos 2\pi x\xi}{\xi}$ is an even function of x, thus the former term equals 0 since f is odd.

Let $\zeta = 2\pi x\xi$, according to (a) we obtain

$$\begin{aligned}
\left| \int_\varepsilon^t \frac{\hat{f}(\xi)}{\xi} \,\mathrm{d}\xi \right| &= \left| \int_{-\infty}^{+\infty} f(x) \left(\int_\varepsilon^t \frac{\sin 2\pi x\xi}{\xi} \,\mathrm{d}\xi \right) \mathrm{d}x \right| \\
&\leq \int_{-\infty}^{+\infty} |f(x)| \left| \int_{2\pi x\varepsilon}^{2\pi x t} \frac{\sin \zeta}{\zeta} \,\mathrm{d}\zeta \right| \mathrm{d}x \\
&= \int_{-\infty}^{+\infty} |f(x)| \,\mathrm{d}x = \parallel f \parallel_1
\end{aligned}$$

□

(c) Let $g(\xi)$ be a continuous odd function that is equal to $1/log(\xi)$ for $\xi \geq 2$. Show that there does not exist an L^1 function whose Fourier transform is g.

Proof. We assume $\exists\, f \in L^1$ such that $\hat{f} = g$.

According to (b), g satisfies

$$\left| \int_2^t \frac{g(\xi)}{\xi} \,\mathrm{d}\xi \right| = \left| \int_2^t \frac{\hat{f}(\xi)}{\xi} \,\mathrm{d}\xi \right| \leq 4 \parallel f \parallel_1, \ \forall\, t > 2$$

Let $\zeta = \log \xi$, however

$$\left| \int_2^t \frac{g(\xi)}{\xi} \,\mathrm{d}\xi \right| = \left| \int_2^t \frac{1}{\xi \log \xi} \,\mathrm{d}\xi \right| = \left| \int_{\log 2}^{\log t} \frac{1}{\zeta} \,\mathrm{d}\zeta \right| = \log\log t - \log\log 2 > \log\log t$$

When $t > e^{e^{4\|f\|_1}}$, there is a contradiction! □

2 Homework 2

2.1 Problem 2.1

Compute

$$\left(\frac{1}{\pi} \mathrm{p.v.} \frac{1}{x} \right)^\wedge (\xi) = -i \mathrm{sgn}(\xi)$$

in the sense of tempered distribution.

Proof. Fixing a $\varphi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned}
\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)^{\wedge}(\varphi) &= \left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)(\hat{\varphi}) \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|\xi|>\varepsilon}\frac{\hat{\varphi}(\xi)}{\xi}\,\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|\xi|>\varepsilon}\frac{1}{\xi}\left(\int_{-\infty}^{+\infty}\varphi(x)e^{-2\pi i x\xi}\,\mathrm{d}x\right)\mathrm{d}\xi \\
&= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\varphi(x)\left(\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\xi}}{\pi\xi}\chi_{\{|x|>\varepsilon\}}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}x
\end{aligned}$$

with Fubini theorem. And according to Fresnel integral, we furthermore have

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{e^{-2\pi i x\xi}}{\pi\xi}\chi_{\{|x|>\varepsilon\}}(\xi)\,\mathrm{d}\xi &= \int_{-\infty}^{+\infty}\frac{\cos 2\pi x\xi - i\sin 2\pi x\xi}{\pi\xi}\chi_{\{|x|>\varepsilon\}}(\xi)\,\mathrm{d}\xi \\
&= -i\int_{-\infty}^{+\infty}\frac{\sin 2\pi x\xi}{\pi\xi}\chi_{\{|x|>\varepsilon\}}(\xi)\,\mathrm{d}\xi \\
&= -i\mathrm{sgn}(x) + 2i\int_0^{\varepsilon}\frac{\sin 2\pi x\xi}{\pi\xi}\,\mathrm{d}\xi
\end{aligned}$$

Substituting back, we obtain

$$\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)^{\wedge}(\varphi) = -i\int_{-\infty}^{+\infty}\varphi(x)\mathrm{sgn}(x)\,\mathrm{d}x + 2i\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\varphi(x)\left(\int_0^{\varepsilon}\frac{\sin 2\pi x\xi}{\pi\xi}\,\mathrm{d}\xi\right)\mathrm{d}x$$

Note that

$$\begin{aligned}
\left|\int_{-\infty}^{+\infty}\varphi(x)\left(\int_0^{\varepsilon}\frac{\sin 2\pi x\xi}{\pi\xi}\,\mathrm{d}\xi\right)\mathrm{d}x\right| &\le \int_{-\infty}^{+\infty}|\varphi(x)|\left|\int_0^{\varepsilon}\frac{\sin 2\pi x\xi}{\pi\xi}\,\mathrm{d}\xi\right|\mathrm{d}x \\
&\le \int_{-\infty}^{+\infty}|\varphi(x)|\int_0^{\varepsilon}\left|\frac{\sin 2\pi x\xi}{\pi\xi}\right|\mathrm{d}\xi\,\mathrm{d}x \\
&\le \varepsilon \parallel \varphi \parallel_1 \\
&\to 0,\ \varepsilon \to 0
\end{aligned}$$

Therefore, the later limit is 0, i.e.

$$\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)^{\wedge}(\varphi) = -i\int_{-\infty}^{+\infty}\varphi(x)\mathrm{sgn}(x)\,\mathrm{d}x,\ \forall\,\varphi\in\mathcal{S}(\mathbb{R})$$

That is to say

$$\left(\frac{1}{\pi}\text{p.v.}\frac{1}{x}\right)^{\wedge}(\xi) = -i\mathrm{sgn}(\xi)$$

in the sense of tempered distribution. □

2.2 Problem 2.2

If $f = \chi_{[0,1]}$, show that $Hf \notin L^1$ and $Hf \notin L^\infty$.

Proof. We firstly calculate Hf. Let $z = x - y$, then

$$Hf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|y|>\varepsilon} \frac{\chi_{[0,1]}(x-y)}{y}\,\mathrm{d}y = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|x-z|>\varepsilon} \frac{\chi_{[0,1]}(z)}{x-z}\,\mathrm{d}z = \begin{cases} -\infty, & x=0 \\ +\infty, & x=1 \end{cases}$$

For $x \neq 0, 1$, when $x \notin (0,1)$ we have

$$Hf(x) = \frac{1}{\pi}\int_0^1 \frac{1}{x-z}\,\mathrm{d}z = \frac{1}{\pi}\log|x| - \frac{1}{\pi}\log|x-1| = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$$

otherwise $x \in (0,1)$, and

$$Hf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{|z-x|>\varepsilon} \frac{\chi_{[0,1]}(z)}{z-x}\,\mathrm{d}z = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$$

To sum up,

$$Hf(x) = \frac{1}{\pi}\log\left|\frac{x}{x-1}\right|$$

Since

$$\int_2^{+\infty} Hf(x) = \frac{1}{\pi}\int_2^{+\infty} \log\left(1 + \frac{1}{x-1}\right)\mathrm{d}x$$

where

$$\log\left(1 + \frac{1}{x-1}\right) \sim \frac{1}{x-1},\ x \to +\infty$$

By comparison discriminant, the integral of $Hf(x)$ on $[2, +\infty)$ diverges. Therefore, $Hf \notin L^1$. In the end, we claim that $Hf \notin L^\infty$. Assuming $Hf \in L^\infty$, there is a null set Z such that

$$\sup_{\mathbb{R}\backslash Z} \log\left|\frac{x}{x-1}\right| < +\infty$$

Consider a family of positive sets

$$E_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)\backslash Z$$

Choose $x_n \in E_n \subset \mathbb{R}\backslash Z$, and note that $x_n \to 1$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} Hf(x_n) = \lim_{x\to 1} Hf(x) = +\infty$$

a contradiction!

In conclusion, we have proved that $f = \chi_{[0,1]} \in L^1 \cap L^\infty$ while $Hf \notin L^1 \cup L^\infty$. □

3 Homework 3

3.1 Problem 3.1

For $\varphi \in \mathcal{S}$, $H\varphi \in L^1$ if and only if $\int \varphi = 0$.

Proof. By definition

$$H\varphi(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{|y|\geq\varepsilon} \frac{\varphi(x-y)}{y}\,\mathrm{d}y$$

Obviously

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 0 \Longleftrightarrow \hat{\varphi}(0) = 0$$

and

$$(H\varphi)^\wedge(\xi) = \frac{1}{\pi}(w_0 * \varphi)^\wedge(\xi) = \frac{1}{\pi}\hat{w}_0(\xi)\hat{\varphi}(\xi) = -i\mathrm{sgn}(\xi)\hat{\varphi}(\xi)$$

where $\hat{\varphi} \in \mathcal{S}$.

$\Longrightarrow$:

Without loss of generality, we assume $\hat{\varphi}(0) = a > 0$ if $\hat{\varphi}(0) \neq 0$. According to the continuity of $\hat{\varphi}$, for $\varepsilon_0 = \frac{a}{2} > 0$, $\exists\,\delta > 0$ such that $\hat{\varphi}(\xi) > \varepsilon_0$ in $(-\delta, \delta)$. In that case, when $0 < |\xi| < \delta$, we have

$$|(H\varphi)^\wedge(\xi)| = |\hat{\varphi}(\xi)| > \varepsilon_0 > 0 = (H\varphi)^\wedge(0)$$

The discontinuity of $(H\varphi)^\wedge$ at 0 implies the fact that $H\varphi \notin L^1$.

$\Longleftarrow$:

Since

$$\left|\frac{\varphi(x-y)-\varphi(x)}{y}\chi_{\{\varepsilon<|y|<1\}}\right| \leq \left|\frac{\varphi(x-y)-\varphi(x)}{y}\right| \leq \max_{[x-1,x+1]} |\varphi'| \leq \|\, \varphi' \,\|_\infty \in L^1[-1,1]$$

we can apply DCT to show that

$$\begin{aligned}
H\varphi(x) &= \lim_{\varepsilon\to 0^+} \int_{|y|>\varepsilon} \frac{\varphi(x-y)}{y}\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|y|<1} \frac{\varphi(x-y)}{y}\,\mathrm{d}y + \int_{|y|\geq 1} \frac{\varphi(x-y)}{y}\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|y|<1} \frac{\varphi(x-y)-\varphi(x)}{y}\,\mathrm{d}y + \int_{|y|\geq 1} \frac{\varphi(x-y)}{y}\,\mathrm{d}y \\
&= \int_{|y|<1} \frac{\varphi(x-y)-\varphi(x)}{y}\,\mathrm{d}y + \int_{|y|\geq 1} \frac{\varphi(x-y)}{y}\,\mathrm{d}y \\
&= I_1(x) + I_2(x)
\end{aligned}$$

We are going to estimate the two integrals respectively.

For the former part, we have similarly that

$$I_1(x) = \frac{\varphi(x-y)-\varphi(x)}{y} \leq \max_{[x-1,x+1]} |\varphi'| \leq \frac{C}{(1+|x|)^2}$$

for sufficiently large $|x|$ since $\varphi \in \mathcal{S}$. Therefore

$$\int_{\mathbb{R}} |I_1(x)|\,\mathrm{d}x = \int_{|x|\leq M} |I_1(x)|\,\mathrm{d}x + \int_{|x|\leq M} |I_1(x)|\,\mathrm{d}x \leq 2M \,\|\, \varphi' \,\|_\infty + \int_{|x|>M} \frac{C}{(1+|x|)^2} < +\infty$$

which implies that $I_1(x) \in L^1$.

For the latter part, we define

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t$$

which is a primitive of $\varphi(x)$. If $\Phi \in L^1$, we can integrate by parts and deduce

$$I_2(x) = \int_{|y|\geq 1} \frac{\varphi(x-y)}{y}\,\mathrm{d}y = \Phi(x+1) - \Phi(x-1) + \int_{|y|\geq 1} \frac{\varphi(x-y)}{y^2}\,\mathrm{d}y$$

which implies

$$\begin{aligned}
\| I_2 \|_1 &\leq 2 \| \Phi \|_1 + \int_{\mathbb{R}} \left(\int_{|y|\geq 1} \frac{\varphi(x-y)}{y^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&\leq 2 \| \Phi \|_1 + \int_{|y|\geq 1} \left(\int_{\mathbb{R}} \frac{\varphi(x-y)}{y^2}\,\mathrm{d}x \right) \mathrm{d}y \\
&\leq 2 \| \Phi \|_1 + 2 \| \varphi \|_1 < +\infty
\end{aligned}$$

In this case, $I_2 \in L^1$, then $H\varphi \in L^1$.

To summarize, we only need to show that $\Phi \in L^1$ to complete the proof. For negative x

$$|\Phi(x)| = \left| \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t \right| = \left| \int_{-\infty}^{x} (1+t)^3 \varphi(t) \frac{1}{(1+t)^3}\,\mathrm{d}t \right| \leq \left| C \int_{-\infty}^{x} \frac{1}{(1+t)^3} \right| \leq \frac{C}{x^2}$$

and for positive x

$$|\Phi(x)| = \left| \int_{-\infty}^{x} \varphi(t)\,\mathrm{d}t \right| = \left| \int_{x}^{+\infty} \varphi(t)\,\mathrm{d}t \right| \leq \left| \int_{x}^{+\infty} \frac{1}{(1+t)^3}\,\mathrm{d}t \right| \leq \frac{C}{x^2}$$

where C is a constant only depending on the Schwartz semi-norm of φ. According to the continuity of Φ at 0, we conclude that $\Phi \in L^1$. □

3.2 Problem 3.2

Check that the Hilbert transform on the line with kernel x^{-1} is a singular integral kernel.

Proof. For $K(x) = \frac{1}{x}$ and dimension $n = 1$ we only need to check three properties of singular integral kernels.

Size Condition: For $B_1 \geq 1$, it is obvious that

$$|K(x)| \leq B_1 |x|^{-1},\ \forall\, x \neq 0$$

Smoothness Condition: According to the symmetrization of $K(x)$, we only need to check this condition for $y > 0$ that

$$\begin{aligned}
\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,\mathrm{d}x &= \int_{-\infty}^{-2y} \frac{|y|}{|x||x-y|}\,\mathrm{d}x + \int_{2y}^{+\infty} \frac{|y|}{|x||x-y|}\,\mathrm{d}x \\
&= y \int_{-\infty}^{-2y} \frac{1}{x(x-y)}\,\mathrm{d}x + y \int_{2y}^{+\infty} \frac{1}{x(x-y)}\,\mathrm{d}x \\
&= y \left(\int_{-\infty}^{+\infty} \frac{1}{x(x-y)}\,\mathrm{d}x - \int_{-2y}^{2y} \frac{1}{x(x-y)}\,\mathrm{d}x \right) \\
&= \ln 3 \leq B_2
\end{aligned}$$

as $B_2 \geq \ln 3$.

Cancellation Condition: It is trivial since $K(x)$ is an odd function on $\mathbb{R}$.

In conclusion, for $B = 2$, the kernel

$$\begin{aligned}
K : \mathbf{R}^n \backslash \{0\} &\longrightarrow \mathbb{R} \\
x &\longmapsto x^{-1}
\end{aligned}$$

is a singular integral kernel. □

3.3 Problem 3.3 (Calderon-Zygmund Decomposition on L^q)

Fix a function $f \in L^q(\mathbb{R}^n)$ for some $1 \le q < +\infty$ and let $\alpha > 0$. Then there exist functions g and b on $\mathbb{R}^n$ such that

1. $f = g + b$.
2. $\| g \|_q \le \| f \|_q$ and $\| g \|_\infty \le 2^{\frac{n}{q}} \alpha$.
3. $b = \sum_j b_j$ where each b_j is supported in a cube Q_j. Furthermore, the cubes Q_j and Q_k have disjoint interiors when $j \ne k$.
4. $\| b_j \|_q^q \le 2^{n+q} \alpha^q |Q_j|$.
5. $\int_{Q_j} b_j = 0$.
6. $\sum_j |Q_j| \le \alpha^{-q} \| f \|_q^q$.
7. $\| b \|_q \le 2^{\frac{n+q}{q}} \| f \|_q$ and $\| b \|_1 \le 2^{\frac{n+q}{q}} \alpha^{1-q} \| f \|_q^q$.

Proof. For $l \in \mathbb{Z}$, we denote D_l as the set of binary cubes whose edge length is 2^l, i.e.

$$D_l = \left\{ \prod_{i=1}^{n} \left[2^l m_i, 2^l(m_i+1)\right) \middle| m_i \in \mathbb{Z} \right\}$$

Obviously, for $Q \in D_l$ and $O' \in D_{l'}$, we have either $Q \cap Q' = \varnothing$ or one of them is the subset of the other.

Since $f \in L^q$, we could find a sufficiently large l_0, such that

$$\left(\frac{1}{|Q|} \int_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le \alpha, \ \forall Q \in D_{l_0}$$

Each $Q \in D_{l_0}$ is composed of 2^n smaller cubes with edge length 2^{l_0-1}. Among these smaller cubes, some satisfy

$$\left(\frac{1}{|Q'|} \int_{Q'} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} > \alpha$$

We put such cubes into a set B, and note that

$$\left(\frac{1}{|Q'|} \int_{Q'} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le \left(\frac{2^n}{|Q|} \int_{Q} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} = 2^{\frac{n}{q}} \left(\frac{1}{|Q|} \int_{Q} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le 2^{\frac{n}{q}} \alpha$$

Others satisfy

$$\left(\frac{1}{|Q'|} \int_{Q'} |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \le \alpha$$

which should be decomposed again.

Step by step, we obtain a set $B = \bigcup_j Q_j$ containing countable binary cubes, and with Lebesgue differentiation theorem, we have

$$f(x) \le \alpha, \ a.e. \ x \notin B$$

Let

$$g(x) = \begin{cases} f(x), & x \notin B \\ \frac{1}{|Q_j|} \int_{Q_j} f(x) \, \mathrm{d}x, & x \in Q_j \end{cases}$$

and

$$b(x) = \sum_j b_j = \sum_j \left(f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(x) \right) \chi_{Q_j}$$

They are well-defined since Hölder inequality implies

$$\frac{1}{|Q_j|} \int_{Q_j} f(x) \,\mathrm{d}x \leq |Q_j|^{\frac{1}{p}-1} \parallel f \parallel_q < +\infty$$

We are going to show they satisfy the given conditions.

Condition 1,3,5 are trivial. **Condition 6** holds since $\forall\, Q' \in B$, we have

$$\begin{aligned}
&\frac{1}{|Q'|} \int_{Q'} |f(x)|^q \,\mathrm{d}x \geq \alpha^q \\
\Longrightarrow &\int_{Q'} |f(x)|^q \,\mathrm{d}x \geq \alpha^q |Q'| \\
\Longrightarrow &\int_{B} |f(x)|^q \,\mathrm{d}x \geq \alpha^q |B| \\
\Longrightarrow &\sum_j |Q_j| \leq \alpha^{-q} \parallel f \parallel_q^q
\end{aligned}$$

Condition 2 holds since

$$\begin{aligned}
\parallel g \parallel_q^q &= \int_{B^c} |f(x)|^q + \sum_j \int_{Q_j} \left| \frac{1}{|Q_j|} \int_{Q_j} f(x) \,\mathrm{d}x \right|^q \mathrm{d}x \\
&= \int_{B^c} |f(x)|^q + \sum_j |Q_j|^{1-q} \left| \int_{Q_j} f(x) \,\mathrm{d}x \right|^q \\
&\leq \int_{B^c} |f(x)|^q + \sum_j \int_{Q_j} |f(x)|^q \,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} |f(x)|^q \,\mathrm{d}x = \parallel f \parallel_q^q
\end{aligned}$$

The inequality comes from Hölder inequality.

For $x \in B^c$, we have shown that $f(x) \leq \alpha, a.e.$; for $x \in Q_j$, we have

$$|g(x)| = \frac{1}{|Q_j|} \left| \int_{Q_j} f(x) \,\mathrm{d}x \right| \leq |Q_j|^{1-\frac{1}{p}} \parallel f \parallel_q = \left(\frac{1}{Q_j} \int_{Q_j} |f(x)|^q \right)^{\frac{1}{q}} \leq 2^{\frac{n}{q}} \alpha$$

Thus, we have shown that

$$\parallel g \parallel_\infty \leq 2^{\frac{n}{q}} \alpha$$

Condition 4 holds since

$$\parallel b_j \parallel_q = \int_{Q_j} \left(f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(x) \,\mathrm{d}x \right)^q \mathrm{d}x \leq 2^{q-1} \left(\int_{Q_j} |f(x)|^q \,\mathrm{d}x + \int_{Q_j} \left| \frac{1}{|Q_j|} \int_{Q_j} f(x) \,\mathrm{d}x \right|^q \mathrm{d}x \right)$$

where

$$\int_{Q_j} |f(x)|^q \,\mathrm{d}x \leq 2^n \alpha^q |Q_j|$$

and

$$\int_{Q_j}\left|\frac{1}{|Q_j|}\int_{Q_j} f(x)\,\mathrm{d}x\right|^q \mathrm{d}x \le |Q_j|^{1-q}\int_{Q_j}|f(x)|^q\,\mathrm{d}x\cdot|Q_j|^{\frac{q}{p}} = \| f \|_q^q \le 2^n\alpha^q|Q_j|$$

Condition 7 holds since

$$\| b \|_1 = \int_B\left|\sum_j b_j\right|\mathrm{d}x \le \sum_j \| b_j \|_1 \le \sum_j |Q_j|^{\frac{1}{p}} \| b_j \|_q \le 2^{\frac{n+q}{q}}\alpha\sum_j|Q_j| \le 2^{\frac{n+q}{q}}\alpha^{1-q} \| f \|_q^q$$

Here we applied Condition 4 and 6. □

4 Homework 4

Let $\mathcal{M}(f)$ and $M(f)$ be the standard centered and uncentered Hardy-Littlewood maximal functions.

4.1 Problem 4.1

Denote the centered Hardy-Littlewood maximal function $\mathcal{M}_c$ and the uncentered Hardy-Littlewood maximal function M_c using cubes with sides parallel to the axes instead of balls in $\mathbb{R}^n$. Prove that

$$\frac{1}{v_n}\frac{2^n}{n^{\frac{n}{2}}} \le \frac{M(f)}{M_c(f)} \le \frac{2^n}{v_n} \qquad \frac{1}{v_n}\frac{2^n}{n^{\frac{n}{2}}} \le \frac{\mathcal{M}(f)}{\mathcal{M}_c(f)} \le \frac{2^n}{v_n}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Conclude that $\mathcal{M}_c$ and M_c are weak type $(1,1)$ and they map $L^p(\mathbb{R}^n)$ to itself for $1 < p \le +\infty$.

Proof. Without loss of generality, we assume the each edge of the cubes is parallel to some axis. Otherwise, we can rotate the coordinates at each given point.

We denote $Q_r(x)$ as the cube centered at x with $2r$-long edges parallel to axes. Direct computations show that

$$\begin{aligned}\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|f(x)|\,\mathrm{d}x &= \frac{1}{v_n r^n}\int_{B_r(x_0)}|f(x)|\,\mathrm{d}x\\ &\le \frac{1}{v_n r^n}\int_{Q_r(x_0)}|f(x)|\,\mathrm{d}x\\ &= \frac{2^n}{v_n|Q_r(x_0)|}\int_{Q_r(x_0)}|f(x)|\,\mathrm{d}x\end{aligned}$$

Therefore,

$$\mathcal{M}(f)(x_0) = \sup_{r>0}\frac{1}{B_r(x_0)}\int_{B_r(x_0)}|f(x)|\,\mathrm{d}x \le \frac{2^n}{v_n}\sup_{r>0}\frac{1}{|Q_r(x_0)|}\int_{Q_r(x_0)}|f(x)|\,\mathrm{d}x = \frac{2^n}{v_n}\mathcal{M}_c(f)(x_0)$$

On the other hand, we deduce through trivial geometric relations that

$$\begin{aligned}\frac{1}{|Q_r(x_0)|}\int_{Q_r(x_0)}|f(x)|\,\mathrm{d}x &= \frac{1}{2^n r^n}\int_{Q_r(x_0)}|f(x)|\,\mathrm{d}x\\ &\le \frac{1}{2^n r^n}\int_{B_{\sqrt{n}r}(x_0)}|f(x)|\,\mathrm{d}x\\ &= \frac{v_n n^{\frac{n}{2}}}{2^n|B_{\sqrt{n}r}(x_0)|}\int_{B_{\sqrt{n}r}(x_0)}|f(x)|\,\mathrm{d}x\end{aligned}$$

Therefore,

$$\mathcal{M}_c(f)(x_0) = \sup_{r>0} \frac{1}{|Q_r(x_0)|} \int_{Q_r(x_0)} |f(x)| \, \mathrm{d}x \le \frac{v_n n^{\frac{n}{2}}}{2^n} \sup_{r>0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |f(x)| \, \mathrm{d}x = \frac{v_n n^{\frac{n}{2}}}{2^n} \mathcal{M}(f)(x_0)$$

So far, we have shown that

$$\frac{1}{v_n} \frac{2^n}{n^{\frac{n}{2}}} \le \frac{\mathcal{M}(f)}{\mathcal{M}_c(f)} \le \frac{2^n}{v_n}$$

Now, we are going to prove the uncentered case. The size relations between balls and cubes are invariant in comparison with the centered case, thus

$$M(f)(x_0) = \sup_{B \ni x_0} \frac{1}{|B|} \int_B |f(x)| \, \mathrm{d}x \le \frac{2^n}{v_n} \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q |f(x)| \, \mathrm{d}x = \frac{2^n}{v_n} M_c(f)(x_0)$$

and

$$M_c(f)(x_0) = \sup_{Q \ni x_0} \frac{1}{|Q|} \int_Q |f(x)| \, \mathrm{d}x \le \frac{v_n n^{\frac{n}{2}}}{2^n} \sup_{B \ni x_0} \frac{1}{|B|} \int_B |f(x)| \, \mathrm{d}x = \frac{v_n n^{\frac{n}{2}}}{2^n} M(f)(x_0)$$

which lead to the conclusion

$$\frac{1}{v_n} \frac{2^n}{n^{\frac{n}{2}}} \le \frac{M(f)}{M_c(f)} \le \frac{2^n}{v_n}$$

Next, we focus on boundedness of the operators.

As is known, standard Hardy-Littlewood maximal operators including the centered and uncentered ones are weak type $(1,1)$ and strong type (p,p) for $p > 1$. Let $p > 1$ and $f \in L^p$, then we obtain

$$\| \mathcal{M}_c(f) \|_p \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| \mathcal{M}(f) \|_p \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| \mathcal{M} \|_{p \to p} \| f \|_p \Longrightarrow \| \mathcal{M}_c \|_{p \to p} \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| \mathcal{M} \|_{p \to p} < +\infty$$

and

$$\| M_c(f) \|_p \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| M(f) \|_p \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| M \|_{p \to p} \| f \|_p \Longrightarrow \| M_c \|_{p \to p} \le \frac{v_n n^{\frac{n}{2}}}{2^n} \| M \|_{p \to p} < +\infty$$

as M and $\mathcal{M}$ are equivalent to each other.

On the other hand, we have

$$m\left(\{x \,|\mathcal{M}_c(f) > \lambda\}\right) \le m\left(\left\{x \,\middle|\, \mathcal{M}(f) > \frac{v_n n^{\frac{n}{2}}}{2^n} \lambda \right\}\right) \le \frac{2^n C_n}{v_n n^{\frac{n}{2}}} \frac{1}{\lambda} \| f \|_1$$

where $f \in L^1$ and C_n is the weak-L^1 norm of $\mathcal{M}$. Similarly, we could show M_c is also weak $(1,1)$. □

4.2 Problem 4.2

Prove that for any fixed $1 < p < +\infty$, the operator norm of M on $L^p(\mathbb{R}^n)$ tends to infinity as $n \to \infty$.

Proof. Abstract an $L^1 \cap L^p$ function

$$f(x) = \chi_{B_1(0)}$$

It is obvious that

$$M(f)(x) = 1, \; \forall \, x \in B_1(0)$$

We construct a ball B_x centered at $\frac{1}{2}(|x| - |x|^{-1})$ with radius $\frac{1}{2}(|x| + |x|^{-1})$. Note that ∂B_x goes through a pair antipodal points of $B_1(0)$, thus

$$|B_x \cap B_1(0)| > \frac{1}{2} B_1(0)$$

With the help of B_x, we can give an estimate of Maximal function for $|x| \geq 1$,

$$M(f)(x) \geq \frac{1}{|B_x|} \int_{B_x} f(x)\, \mathrm{d}x = \frac{|B_x \cap B_1(0)|}{|B_x|} \geq 2^{n-1}(|x| + |x|^{-1})^{-n}$$

Direct computation yields

$$\begin{aligned}
\| M(f) \|_p^p &= \int_{B_1(0)} |f(x)|^p \, \mathrm{d}x + \int_{B_1(0)^c} |f(x)|^p \, \mathrm{d}x \\
&\geq |B_1(x)| + \int_{\mathbb{R}^n} 2^{np-p}(|x| + |x|^{-1})^{-np} \, \mathrm{d}x \\
&= |B_1(x)| + |\partial B_1(0)| 2^{np-p} \int_1^{+\infty} \frac{r^{n-1}}{(r + \frac{1}{r})^{np}}
\end{aligned}$$

We have obtain an estimate of (p, p) norm of M for fixed p, which is

$$\| M \|_{p\to p}^p \geq \frac{\| Mf \|_p}{\| f \|_p^p} = 1 + \frac{|\partial B_1(0)|}{|B_1(0)|} 2^{np-p} \int_1^{+\infty} \frac{r^{n-1}}{(r + \frac{1}{r})^{np}} = 1 + 2^{np-p} n \int_1^{+\infty} \frac{r^{n-1}}{(r + \frac{1}{r})^{np}}$$

Therefore, we only need to show that

$$\lim_{n\to\infty} 2^{np-p} n \int_1^{+\infty} \frac{r^{n-1}}{(r + \frac{1}{r})^{np}} = +\infty$$

for given p.

It is easy to verify that $\forall\, N > 1$, $\exists\, r_0 > 1$, such that

$$r_0 + \frac{1}{r_0} = N r_0$$

and

$$r + \frac{1}{r} \leq Nr, \ \forall\, r \geq r_0$$

then we have

$$\begin{aligned}
2^{np-p} n \int_1^{+\infty} \frac{r^{n-1}}{(r + \frac{1}{r})^{np}} &\geq 2^{np-p} n \int_1^{+\infty} \frac{r^{n-1}}{N^{np} r^{np}} \, \mathrm{d}r \\
&\geq \frac{2^{np-p} n}{N^{np}} \int_{r_0}^{+\infty} \frac{1}{r^{np-n+1}} \, \mathrm{d}r \\
&= \frac{2^{np-p}}{N^{np}(p-1)} \frac{1}{r_0^{np-n}} \\
&= \frac{1}{2^p(p-1)} \frac{2^{np} r_0^n}{(N r_0)^{np}} \\
&= \frac{1}{2^p(p-1)} \left(\frac{2^p r_0}{(N r_0)^p} \right)^n \\
&\to +\infty, \ n \to \infty
\end{aligned}$$

as long as

$$\frac{2^p r_0}{(Nr_0)^p} > 1 \Longleftrightarrow 2r_0^{\frac{1}{p}} > Nr_0 = r_0 + \frac{1}{r_0}$$

We will reach the final conclusion by explaining the existence of such an r_0. Abstract

$$f(t) = 2t^{\frac{1}{p}} - t - \frac{1}{t}$$
$$f'(t) = \frac{2}{p} t^{\frac{1-p}{p}} - 1 + \frac{1}{t^2}$$

and we note that

$$f(1) = 0 \qquad f'(1) = \frac{2}{p} > 0$$

Therefore, there is a small δ and $r_0 \in (1, 1+\delta)$ such that $f(r_0) > 0$. The proof is finished. □

5 Homework 5

5.1 Problem 5.1

Find an example showing that the product of two BMO functions may not be in BMO.

Proof. We have proved that $f(x) = \log|x|$ is a BMO function. **Problem 5.3**, however, shows the fact that $(f(x))^2 = |\log|x||^2$ is not in BMO. Thus, we only need to prove **Problem 5.3** below. □

5.2 Problem 5.2

Prove that

$$\| \, |f|^\alpha \, \|_* \leq 2 \, \| \, f \, \|_*^\alpha$$

whenever $0 < \alpha < 1$.

Proof. Since $\alpha > 0$, we have the inequality

$$\left|\frac{x_1}{x_1+x_2}\right|^\alpha + \left|\frac{x_2}{x_1+x_2}\right|^\alpha \leq 1 \Longrightarrow |x_1+x_2|^\alpha \leq |x_1|^\alpha + |x_2|^\alpha \Longrightarrow |x_1|^\alpha - |x_2|^\alpha \leq |x_1 - x_2|^\alpha$$

Therefore, we obtain

$$\int_Q \left||f|^\alpha - |f_Q|^\alpha\right| \leq \int_Q |f - f_Q|^\alpha$$

Let $p = \frac{1}{\alpha} \geq 1$. According to Hölder's inequality, we have for any given cube Q that

$$\frac{1}{|Q|}\int_Q \left||f|^\alpha - |f_Q|^\alpha\right| = \frac{1}{|Q|}\int_Q \left|f - f_Q\right|^\alpha \leq \frac{1}{|Q|}\left(\int_Q \mathrm{d}x\right)^{\frac{1}{p'}} \left(\int_Q (f - f_Q)^p\right)^{\frac{1}{p}} = \left(\frac{1}{|Q|}\int_Q (f - f_Q)\right)^\alpha$$

Additionally, we apply triangle inequality

$$\left||f|^\alpha - (|f|^\alpha)_Q\right| \leq \left||f|^\alpha - |f_Q|^\alpha\right| + \left||f_Q|^\alpha - (|f|^\alpha)_Q\right| \leq \left||f|^\alpha - |f_Q|^\alpha\right| + \frac{1}{|Q|}\int_Q \left|f - (|f|^\alpha)_Q\right|$$

Therefore,

$$\| \, |f|^\alpha \, \|_* = \sup_Q \int_Q \left(|f|^\alpha - (|f|^\alpha)_Q\right) \leq 2 \sup_Q \int_Q \left||f|^\alpha - |f_Q|^\alpha\right| \leq 2 \sup_Q \left(\frac{1}{|Q|}\int_Q (f - f_Q)\right)^\alpha \leq 2 \, \| \, f \, \|_*^\alpha$$

□

5.3 Problem 5.3

Prove that $|\log|x||^p$ is not in $BMO(\mathbb{R})$ when $1 < p < +\infty$.

Proof. If $f(x) = |\log|x||^p \in BMO(\mathbb{R})$ for $p > 1$, John-Nirenberg inequality implies

$$\sup_Q \frac{1}{|Q|} \int_Q e^{\frac{C_2}{\|f\|_*}|f(x)-f_Q|} \leq C_1 \tag{1}$$

That is to say

$$e^{\frac{C_2}{\|f\|_*}|f(x)-f_Q|} \in L^1_{loc}(\mathbb{R})$$

However, in a small neighborhood of 0, say $(-\delta, \delta)$, we have

$$\frac{C_2}{\| f \|_*} |\ln|x||^{p-1} > 2$$

as long as δ is sufficiently small. Therefore, for this fixed δ, we have

$$e^{c|f(x)-f_Q|} \geq e^{-cf_Q} e^{c|\ln|x||^p} = C_Q \left(\frac{1}{|x|}\right)^{c|\ln|x||^{p-1}} > C_Q \left(\frac{1}{Q}\right) = \frac{C_Q}{|x|^2}$$

which is not L^1 near 0. In other words, we have

$$\sup_Q \frac{1}{|Q|} \int_Q e^{\frac{C_2}{\|f\|_*}|f(x)-f_Q|} \geq \frac{1}{2\delta} \int_{-\delta}^{\delta} e^{\frac{C_2}{\|f\|_*}|f(x)-f_Q|} = +\infty$$

A contradiction! □

6 Homework 6

6.1 Problem 6.1

Construct a Schwartz function Ψ that satisfies

$$\sum_{j \in \mathbb{Z}} \left| \hat{\Psi}(2^{-j}\xi) \right|^2 = 1$$

for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and whose Fourier transform is supported in the annulus $\frac{6}{7} \leq |\xi| \leq 2$ and is equal to 1 on the annulus $1 \leq |\xi| \leq \frac{13}{7}$.

Proof. We firstly construct a radial C_c^∞ function ψ such that

$$\hat{\psi}|_{[\frac{29}{30}, \frac{19}{10}]} = 1 \qquad supp\, \hat{\psi} \subset \left[\frac{19}{20}, \frac{39}{20}\right] \qquad 0 \leq \hat{\psi} \leq 1$$

Obviously, $\hat{\psi}^2$ also satisfies the conditions above.

Note that

$$\frac{39}{40} < 1 < \frac{13}{7} < \frac{19}{10}$$

thus for $\xi \in [1, \frac{13}{7}]$

$$\hat{\psi}^2 \left(2^{-j}\xi\right) \neq 0 \Longleftrightarrow j = 0$$

Moreover,

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j(\xi)\right|^2=\left|\hat{\psi}(\xi)\right|^2,\ \xi\in\left[1,\frac{13}{7}\right]$$

for $\hat{\psi}_j(\xi)=\hat{\psi}(2^{-j}\xi)$. The left sum is well-defined since each ξ only located in finitely many supports of $\{\hat{\psi}_j\}$.

It is easy to verify the fact that

$$supp\,\hat{\psi}_j\cap supp\,\hat{\psi}_{j+1}=\left[\frac{19}{20}2^j,\frac{39}{20}2^j\right]\cap\left[\frac{19}{20}2^{j+1},\frac{39}{20}2^{j+1}\right]=\left[\frac{38}{20}2^j,\frac{39}{20}2^j\right]\neq\varnothing$$

and $\forall\,\xi\in\mathbb{R}^n\backslash\{0\}$, $\exists\,j$, such that $\xi\in supp\,\hat{\psi}$. Therefore,

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j(\xi)\right|^2>0,\ \forall\,\xi\in\mathbb{R}^n\backslash\{0\}$$

Now, we can construct

$$\hat{\Psi}(\xi)=\left(\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j(\xi)\right|^2\right)^{-\frac{1}{2}}\hat{\psi}(\xi)\in\mathbb{S}$$

that satisfies all requirements in the problem. Therefore, we can let

$$\Psi(\xi)=\left(\left(\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j(\xi)\right|^2\right)^{-\frac{1}{2}}\hat{\psi}(\xi)\right)^{\vee}=\left(\left(\sum_{j\in\mathbb{Z}}\left|\hat{\psi}_j(\xi)\right|^2\right)^{-\frac{1}{2}}\right)^{\vee}*\psi(x)\in\mathcal{S}$$

□

6.2 Problem 6.2

Suppose that $\varphi(\xi)$ is a smooth function on $\mathbb{R}^n$ that vanishes in a neighborhood of the origin and is equal to 1 in a neighborhood of infinity. Prove that the function $e^{-i\xi_j|\xi|^{-1}}\varphi(\xi)$ is in $\mathscr{M}_p(\mathbb{R}^n)$ for $1<p<+\infty$.

Proof. According to Mikhlin theorem, we only need to show

$$\left|D^\gamma\left(e^{i\xi_j|\xi|^{-1}}\varphi(\xi)\right)\right|\leq B|\xi|^{-|\gamma|},\ \forall\,\xi\neq 0$$

for multi-index γ such that $|\gamma|\leq n+2$.

In fact

$$\left|D^\gamma\left(e^{i\xi_j|\xi|^{-1}}\varphi(\xi)\right)\right|=\sum_{\alpha\leq\gamma}C_{\alpha,\gamma}\left|D^\alpha\left(e^{i\xi_j|\xi|^{-1}}\right)\right|\left|D^{\gamma-\alpha}\varphi(\xi)\right|$$

Here $\alpha<\gamma$ means each component of α is less than or equal to the corresponding component of γ.

Note that $1-\varphi$ is a standard bump function, as is known

$$\left|D^{\gamma-\alpha}\varphi(\xi)\right|\leq\frac{C_{\gamma-\alpha}}{|\xi|^{|\gamma-\alpha|}},\ |\gamma-\alpha|\geq 1$$

On the other hand, for $k\neq j$

$$\frac{\partial}{\partial\xi_j}e^{i\xi_j|\xi|}=e^{i\xi_j|\xi|}\left(\frac{1}{|\xi|}-\frac{\xi_j^2}{|\xi|^3}\right)$$
$$\frac{\partial}{\partial\xi_k}e^{i\xi_j|\xi|}=-e^{i\xi_j|\xi|}\frac{\xi_j\xi_k}{|\xi|^3}$$

thus

$$\left|D^\alpha\left(e^{i\xi_j|\xi|}\right)\right| \le \frac{C_\alpha}{|\xi|},\ |\alpha| = 1$$

Moreover, we can prove by induction that

$$\left|D^\alpha\left(e^{i\xi_j|\xi|}\right)\right| \le \frac{C_\alpha}{|\xi|},\ \alpha \le \gamma$$

Since α and γ only take finitely many values, there is a global constant C depending only on n and φ, such that

$$\left|D^\gamma\left(e^{i\xi_j|\xi|^{-1}}\varphi(\xi)\right)\right| \le C\sum_{\alpha\le\gamma}\frac{1}{|\xi|^{|\gamma|}}\frac{1}{|\xi|^{\gamma-\alpha}} = \frac{C}{|\xi|^{|\gamma|}}$$

which implies

$$e^{-i\xi_j|\xi|^{-1}}\varphi(\xi) \in \mathscr{M}_p(\mathbb{R}^n)$$

□

7 Homework 7

7.1 Problem 7.1

(Muscalu, Schlag, *Classical and Multilinear Harmonic Analysis*, Vol.1, Section 8.2, Corollary 8.4 (i), P202-203) In their proof of Corollary 8.4 (i), they write

$$S(f_k - f)(x) \le \varliminf_{m\to\infty}(f_k - f_m)(x)$$

Please provide a proof of this inequality.

Proof. In their proof, it is supposed that $f_m \to f$ in L^p, where $f_k \in \mathcal{S}, f \in L^p$. Therefore, we have

$$S(f_k - f) = \left(\sum_{j\in\mathbb{Z}}|P_j(f_k - f)|^2\right)^{\frac{1}{2}}$$

Since

$$\| P_j f \|_p = \| \check{\psi}_j * f \|_p \le \| \check{\psi}_j \|_1 \| f \|_p < +\infty$$

we confirm that $P_j f$ is well-defined.

Additionally, we note that

$$|P_j f_m - P_j f|_1 = |\check{\psi}_j * (f_m - f)| \le \| \check{\psi}_j \|_q \| f_m - f \|_p \to 0,\ m\to\infty$$

which implies

$$\lim_{m\to\infty} P_j f_m = P_j f$$

Therefore, we have with Fatou's lemma that

$$\begin{aligned}
S(f_k - f) &= \left(\sum_{j\in\mathbb{Z}}|P_j(f_k - f)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j\in\mathbb{Z}}\lim_{m\to\infty}|P_j(f_k - f_m)|^2\right)^{\frac{1}{2}} \\
&\le \varliminf_{m\to\infty}\left(\sum_{j\in\mathbb{Z}}|P_j(f_k - f)|^2\right)^{\frac{1}{2}} \\
&= \varliminf_{m\to\infty}(f_k - f_m)
\end{aligned}$$

$\square$

7.2 Problem 7.2(Khinchins inequality)

In its proof, one can first prove the following version with real coefficients a_n,

$$\mathbb{E}\left(\left|\sum_{n=1}^{N} a_n\omega_n\right|^p\right)^{\frac{1}{p}} \asymp \left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac{1}{2}}$$

for $1 < p < +\infty$. Assuming the above inequality, one can then extend it to a version with complex coefficients a_n. Please explain how to extend it to complex coefficients.

Proof. Let $z_n = x_n + iy_n$ be a complex number, where x_n and y_n are real.

As is proved, we have

$$c_1\left(\sum_{n=1}^{N}|x_n|^2\right)^{\frac{1}{2}} \leq \mathbb{E}\left(\left|\sum_{n=1}^{N} x_n\omega_n\right|^p\right)^{\frac{1}{p}} \leq c_2\left(\sum_{n=1}^{N}|x_n|^2\right)^{\frac{1}{2}}$$
$$c_1\left(\sum_{n=1}^{N}|y_n|^2\right)^{\frac{1}{2}} \leq \mathbb{E}\left(\left|\sum_{n=1}^{N} y_n\omega_n\right|^p\right)^{\frac{1}{p}} \leq c_2\left(\sum_{n=1}^{N}|y_n|^2\right)^{\frac{1}{2}}$$

For $\{z_n\}$, we apply Minkovski inequality

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{n=1}^{N} z_n\omega_n\right|^p\right)^{\frac{1}{p}} &= \mathbb{E}\left(\left|\sum_{n=1}^{N} x_n\omega_n + iy_n\omega_n\right|^p\right) \\
&\leq \left(\int_\Omega\left(\left|\sum_{n=1}^{N} x_n\omega_n\right| + \left|\sum_{n=1}^{N} y_n\omega_n\right|\right)^p \mathrm{d}P\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega\left|\sum_{n=1}^{N} x_n\omega_n\right|^p \mathrm{d}P\right)^{\frac{1}{p}} + \left(\int_\Omega\left|\sum_{n=1}^{N} y_n\omega_n\right|^p \mathrm{d}P\right)^{\frac{1}{p}} \\
&\leq c_2\left(\sum_{n=1}^{N}|x_n|^2\right)^{\frac{1}{2}} + c_2\left(\sum_{n=1}^{N}|y_n|^2\right)^{\frac{1}{2}} \\
&\leq 2c_2\left(\sum_{n=1}^{N}|z_n|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left(\left|\sum_{n=1}^{N} z_n\omega_n\right|^p\right)^{\frac{1}{p}} &\geq \frac{1}{2}\left(\int_\Omega\left|\sum_{n=1}^{N} x_n\omega_n\right|^p \mathrm{d}P\right)^{\frac{1}{p}} + \frac{1}{2}\left(\int_\Omega\left|\sum_{n=1}^{N} y_n\omega_n\right|^p \mathrm{d}P\right)^{\frac{1}{p}} \\
&\geq \frac{c_1}{2}\left(\sum_{n=1}^{N}|x_n|^2\right)^{\frac{1}{2}} + \frac{c_1}{2}\left(\sum_{n=1}^{N}|y_n|^2\right)^{\frac{1}{2}} \\
&\geq \frac{c_1}{2}\left(\sum_{n=1}^{N}(|x_n| + |y_n|)^2\right)^{\frac{1}{2}} \\
&\geq \frac{c_1}{2}\left(\sum_{n=1}^{N}|z_n|^2\right)^{\frac{1}{2}}
\end{aligned}$$

In this proof, we utilized the relation

$$\max\{|x_n|, |y_n|\} \leq |z_n| \leq |x_n| + |y_n|$$

□

8 Homework 8

8.1 Problem 8.1

Fix a nonzero Schwartz function h on the line whose Fourier transform is supported in the interval $[-\frac{1}{8}, \frac{1}{8}]$. For $\{a_j\}$, a sequence of numbers, set

$$f(x) = \sum_{j=1}^{\infty} a_j e^{2\pi i 2^j x} h(x)$$

Prove that for all $1 < p < +\infty$ there exists a constant C_p such that

$$\| f \|_{L^p} \leq C_p \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \| h \|_{L^p}$$

Proof. Abstract $\varphi \in C_c^\infty$ such that

$$\varphi|_{[\frac{7}{8}, \frac{9}{8}]} = 1 \qquad \varphi|_{[\frac{3}{4}, \frac{5}{4}]^c} = 0 \qquad 0 \leq \varphi \leq 1$$

Similar to Littlewood-Paley square function, we define T_j such that

$$T_j f = \left(\varphi_j \hat{f} \right)^\vee = \check{\varphi}_j * f$$

where $\varphi_j(x) = \varphi(2^{-j}x)$, and T_j is bounded since

$$\| T_j f \|_p \leq \| \varphi_j \|_1 \| f \|_p = C \| f \|_p$$

according to Young's inequality.

Next, we claim an estimate

$$f(x) = \sum_{j=1}^{\infty} T_j \left(a_j e^{2\pi i 2^j x} h(x) \right)$$

Actually, the translation property of Fourier transform shows that

$$\left(a_j e^{2\pi i 2^j x} h(x) \right)^\wedge = a_j \hat{h}(\xi - 2^j) = a_j \varphi_j \hat{h}(\xi - 2^j) = \left(T_j \left(a_j e^{2\pi i 2^j x} h(x) \right) \right)^\wedge$$

so

$$\hat{f}(\xi) = \left(\sum_{j=1}^{\infty} T_j \left(a_j e^{2\pi i 2^j x} h(x) \right) \right)^\wedge$$

The smoothness of φ_j implies the smoothness of f, thus the claim is proved after an inverse transform.

Littlewood-Paley theorem shows that

$$\| f \|_p = \left\| \sum_{j=1}^{\infty} T_j \left(a_j e^{2\pi i 2^j x} h(x) \right) \right\|_p \leq C_p \left\| \left(\sum_{j=1}^{\infty} T_j \left| a_j e^{2\pi i 2^j x} h(x) \right|^2 \right)^{\frac{1}{2}} \right\|_p = C_p \left(\sum_{j=1}^{\infty} |a_j|^2 \right)^{\frac{1}{2}} \| h \|_p$$

□

8.2 Problem 8.2

(The Homework on P.27 Chapter 4, Wolff) Using translation and multiplication by characters, construct a sequence of Schwartz functions $\{\varphi_n\}$ so that

1. Each φ_n has the same L^p norm.
2. Each $\hat{\varphi}_n$ has the same $L^{p'}$ norm.
3. The supports of the $\hat{\varphi}_n$ are disjoint.
4. The supports of the $\{\varphi_n\}$ are "essentially disjoint" meaning that

$$\left\|\sum_{n=1}^{N}\varphi_n\right\|_p^p \approx \sum_{n=1}^{N} \| \varphi_n \|_p^p \approx N$$

   uniformly in N.

Proof. Let $\varphi \in \mathcal{S}$ satisfies

$$supp\,\hat{\psi} \subset B_1(0)$$

We will consider its translations.

Abstract

$$\hat{\psi}_k(\xi) = \hat{\psi}(\xi + 3ke_n) \in \mathcal{S}$$

which is supported in $B_1(3ke_n)$. Therefore, the supports of $\{\hat{\psi}_k\}$ are disjoint.

Obviously, the $L^{p'}$ norm of $\hat{\psi}_k$ does not rely on k. Additionally,

$$\psi_k(x) = e^{-6\pi i k x_n}\psi(x) \Longrightarrow |\psi_k(x)| = |\psi(x)| \Longrightarrow \| \psi_k \|_p = \| \psi \|_p$$

So far, condition 1,2,3 have been satisfied.

Similarly, an additional translation

$$\varphi_k(x) = \psi(x + h_k)$$

does not violate condition 1 and 2. Therefore, we only need to select appropriate $\{h_k\}$ such that condition 3 and 4 hold.

It is trivial that

$$\sum_{k=1}^{N} \| \varphi_k \|_p^p \approx N$$

since $\varphi_k \in \mathcal{S}$.

When $N = 1$, $h_1 = 0$ is suitable.

When $N = 2$,

$$\| \varphi_1 + \varphi_2 \|_p^p = \int_{\mathbb{R}^n} \left| e^{-6\pi i x_n}\psi(x) + e^{-12\pi i x_n}\psi(x - h_2)\right|^p \mathrm{d}x$$

Since $\varphi_k \in \mathcal{S}$, they "almost vanish" outside a small compact set. we can find a sufficiently large $|h_2|$, such that

$$\| \varphi_1 + \varphi_2 \|_p^p \approx \| \varphi_1 \|_p^p + \| \varphi_2 \|_p^p$$

and

$$supp\,\hat{\varphi}_1 \cap supp\,\hat{\varphi}_2 = \varnothing$$

For larger N, we can select h_k one by one to satisfy condition 3 and 4, and this process is well-defined since N is a finite number. ☐

9 Homework 9

9.1 Problem 9.1

Suppose that φ is a real C^∞ phase function satisfying the non-degeneracy condition

$$\det\left(\frac{\partial^2\varphi}{\partial x_j \partial y_k}\right) \neq 0$$

on the support of $a(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then for $\lambda > 0$,

$$\left\|\int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)} a(x,y) f(y)\,\mathrm{d}y\right\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-\frac{n}{2}} \parallel f \parallel_{L^2(\mathbb{R}^n)}$$

Proof. Abstract linear operator

$$\begin{aligned} T : L^2(\mathbb{R}^n) &\to L^2(\mathbb{R}^n) \\ f(x) &\to \int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)} a(x,y) f(y)\,\mathrm{d}y \end{aligned}$$

We are going to focus on its L^2 boundedness.

In order to obtain the operator norm of T, we need to compute its adjoint operator T^*. By Fubini theorem

$$\begin{aligned} (Tf, g)_{L^2} &= \int_{\mathbb{R}^n} \overline{g(x)} \left(\int_{\mathbb{R}^n} e^{i\lambda\varphi(x,y)} a(x,y) f(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(y) \overline{\left(\int_{\mathbb{R}^n} e^{-i\lambda\varphi(x,y)} \overline{a(x,y)} g(x)\,\mathrm{d}x\right)}\,\mathrm{d}y \\ &= (f, T^*g)_{L^2} \end{aligned}$$

where

$$T^*f(x) = \int_{\mathbb{R}^n} e^{-i\lambda\varphi(y,x)} \overline{a(y,x)} f(y)\,\mathrm{d}y$$

We only need to show that

$$\parallel S \parallel = \parallel TT^* \parallel = \parallel T \parallel^2 \leq C\lambda^{-n}$$

where $S = TT^*$

Direct computation shows

$$\begin{aligned} Sf(x) &= \int_{\mathbb{R}^n} e^{i\lambda\varphi(x,t)} a(x,t) \left(\int_{\mathbb{R}^n} e^{-i\lambda\varphi(s,t)} \overline{a(s,t)} f(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i\lambda(\varphi(x,t)-\varphi(s,t))} a(x,t)\overline{a(s,t)} f(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{i\lambda(\varphi(x,t)-\varphi(s,t))} a(x,t)\overline{a(s,t)}\,\mathrm{d}t\right) f(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n} K(x,s) f(s)\,\mathrm{d}s \end{aligned}$$

where K is a smooth function.

For a pending vector ν, we consider the directional derivative of t in ν, that is

$$\begin{aligned}\frac{\partial}{\partial \nu}\left(\varphi(x,t)-\varphi(s,t)\right) &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial t_i}\varphi(x,t)-\frac{\partial}{\partial t_i}\varphi(s,t)\right)\nu_i \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial^2\varphi}{\partial t_i \partial t_j}(x_j-s_j)+O(|x-s|^2)\right)\nu_i \\ &= (x-s)D_t^2\varphi(x,t)\nu^T\end{aligned}$$

Here

$$D_t^2\varphi(x,t)=\begin{pmatrix}\frac{\partial^2\varphi}{\partial t_1^2}(x,t) & \frac{\partial^2\varphi}{\partial t_1\partial t_2}(x,t) & \cdots & \frac{\partial^2\varphi}{\partial t_1\partial t_n}(x,t)\\ \frac{\partial^2\varphi}{\partial t_2\partial t_1}(x,t) & \frac{\partial^2\varphi}{\partial t_2^2}(x,t) & \cdots & \frac{\partial^2\varphi}{\partial t_2\partial t_n}(x,t)\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2\varphi}{\partial t_n\partial t_1}(x,t) & \frac{\partial^2\varphi}{\partial t_n\partial t_2}(x,t) & \cdots & \frac{\partial^2\varphi}{\partial t_n^2}(x,t)\end{pmatrix}$$

is a Hessian matrix.

Let

$$\nu^T=\frac{1}{|x-t|}\left(D_t^2\varphi(x,t)\right)^{-1}(x-t)^T$$

and we have

$$\frac{\partial}{\partial \nu}\left(\varphi(x,t)-\varphi(s,t)\right)=|x-s|+O(|x-s|^2)$$

Since a is compactly supported, there is a uniform constant C, such that

$$\frac{\partial}{\partial \nu}\left(\varphi(x,t)-\varphi(s,t)\right)\geq C|x-s|$$

on its support.

Now we define a differential operator

$$\tilde{D}:f\to\left(\frac{\partial f}{\partial \nu}(t)\right)\left(i\lambda\frac{\partial}{\partial \nu}\left(\varphi(x,t)-\varphi(s,t)\right)\right)^{-1}$$

which satisfies

$$\tilde{D}\left(e^{i\lambda(\varphi(x,t)-\varphi(s,t))}\right)=e^{i\lambda(\varphi(x,t)-\varphi(s,t))}$$

Moreover, we denote

$$\tilde{D}^{\tau}:f\to-\tilde{D}\left(\frac{f(t)}{i\lambda\frac{\partial}{\partial\nu}(\varphi(x,t)-\varphi(s,t))}\right)$$

Now $\forall\,\varepsilon>0$, we have

$$|K(x,s)|\leq C(\varepsilon),\ |x-s|<\varepsilon$$

and

$$|K(x,s)|=\left|\int_{\mathbb{R}^n}e^{i\lambda(\varphi(x,t)-\varphi(s,t))}(\tilde{D}^{\tau})^N(a(x,t)\overline{a(s,t)})\,\mathrm{d}t\right|\leq\frac{C(\varepsilon)}{\lambda^N|x-s|^N},\ |x-s|\geq\varepsilon$$

resulted from integrating by parts N times.

For sufficiently small ε such that

$$\varepsilon<\frac{1}{2}diam(supp\,a)$$

we have

$$|K(x,s)|\leq\frac{C_N}{(1+\lambda|x-s|)^N}$$

To sum up, we obtain the estimate by selecting $N > n$

$$\| S \| \leq C \int_{\mathbb{R}^n} \frac{\mathrm{d}x}{(1+\lambda|x|)^m} = C \int_0^{+\infty} \frac{r^{n-1}\,\mathrm{d}r}{(1+\lambda r)^N} \leq C\lambda^{-n}$$

□

9.2 Problem 9.2

Prove the identity

$$\int_0^{+\infty} e^{i\lambda x^k} e^{-x^k} x^l \,\mathrm{d}x = c_{k,l}(1-i\lambda)^{-\frac{l+1}{k}} \tag{2}$$

for any integers $k \geq 2$ and $l \geq 0$, where $c_{k,l}$ is constant.

Proof. Let $t = (1-i\lambda)^{\frac{1}{k}}$ be an constant, then the identity is converted into

$$\int_0^{+\infty} e^{-t^k x^k} x^l \,\mathrm{d}x = c_{k,l} t^{-(l+1)}$$

thus we only need to show

$$\int_0^{+\infty} e^{-t^k x^k} t^{l+1} x^l \,\mathrm{d}x$$

is independent of λ.

We change variable by $y = tx$, then

$$\int_0^{+\infty} e^{-t^k x^k} t^{l+1} x^l \,\mathrm{d}x = \int_0^{+\infty} e^{-y^k} y^l \,\mathrm{d}y = c_{k,l}$$

The identity above is established by Cauchy integration theorem since the integrand is a Schwartz function whose integral vanished at infinity. □

9.3 Problem 9.3

Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies that

$$\varphi(x_0) = \varphi'(x_0) = \varphi''(x_0) = 0$$

while $\varphi'''(x_0) \neq 0$. If ψ is supported in a sufficiently small neighborhood of x_0, prove that

$$\int_{\mathbb{R}} e^{i\lambda\varphi(x)} \psi(x) \,\mathrm{d}x = \lambda^{-\frac{1}{3}} \sum_{j=1}^{N} a_j \lambda^{-\frac{j}{3}} + o\left(\lambda^{-\frac{N+2}{3}}\right)$$

for all $\lambda > 1$ and nonnegative integer N.

Proof. We first consider $\varphi(x) = x^3$ for $x_0 = 0$,

$$I(\lambda) = \int_{\mathbb{R}} e^{i\lambda x^3} \psi(x) \,\mathrm{d}x = \int_{\mathbb{R}} e^{(i\lambda-1)x^3} e^{x^3} \psi(x) \,\mathrm{d}x$$

Abstract the Taylor expansion

$$e^{x^3}\psi(x) = \sum_{j=0}^{N} a_j x^j + x^{N+1} R_N(x)$$

Plugging it into the integral, we obtain

$$I(\lambda) = \sum_{j=0}^{N} \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^j \,\mathrm{d}x + \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \,\mathrm{d}x$$

where

$$\begin{aligned}
\int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^j \,\mathrm{d}x &= (1-i\lambda)^{-\frac{j+1}{3}} \int_{\mathbb{R}} e^{-'y^3} y^j \,\mathrm{d}x \\
&= \lambda^{-\frac{j+1}{3}} (\lambda^{-1}-i)^{-\frac{j+1}{3}} \int_{\mathbb{R}} e^{-'y^3} y^j \,\mathrm{d}x \\
&= \lambda^{-\frac{j+1}{3}} \int_{\mathbb{R}} e^{-'y^3} y^j \,\mathrm{d}x \left(\sum_{l=0}^{L} C_{j,l} \lambda^{-l} + O(\lambda^{-l-1}) \right)
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \,\mathrm{d}x = \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \alpha\left(\frac{x}{\varepsilon}\right) \mathrm{d}x + \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \left(1-\alpha\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x$$

here $\alpha(x)$ is a cut off function supported in $[-2,2]$ and reaches value 1 in $[-1,1]$.

Therefore, the former term

$$\begin{aligned}
|I_1| &= \left| \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \alpha\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \right| \\
&= \left| \int_{-2\varepsilon}^{2\varepsilon} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \alpha\left(\frac{x}{\varepsilon}\right) \mathrm{d}x \right| \\
&\le \int_{-2\varepsilon}^{2\varepsilon} |x|^{N+1} \left| e^{(i\lambda-1)x^3} R_N(x) \right| \mathrm{d}x \\
&\le C\varepsilon^{N+2}
\end{aligned}$$

and the latter term

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x) \left(1-\alpha\left(\frac{x}{\varepsilon}\right)\right) \mathrm{d}x \\
&= \frac{1}{i\lambda} \left(\frac{x^{N+1} e^{-x^3} R_N(x)(1-\alpha(\frac{x}{\varepsilon}))}{3x^2} \right) \Bigg|_{-\infty}^{+\infty} \\
&\quad - \int_{\mathbb{R}} e^{-i\lambda x^3} \left(\frac{x^{N+1} e^{-x^3} R_N(x)(1-\alpha(\frac{x}{\varepsilon}))}{3x^2} \right)' \mathrm{d}x \\
&= - \int_{\mathbb{R}} e^{-i\lambda x^3} \left(\frac{x^{N+1} e^{-x^3} R_N(x)(1-\alpha(\frac{x}{\varepsilon}))}{3x^2} \right)' \mathrm{d}x
\end{aligned}$$

Now, we define a differential operator

$$Df = \frac{1}{3i\lambda x^2} \frac{\mathrm{d}f}{\mathrm{d}x}$$

and additionally

$$D_\tau f = -\frac{1}{i\lambda} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{f}{3x} \right)$$

then

$$I_2 = \int_{\mathbb{R}} D^M \left(e^{i\lambda x^3}\right) \left(x^{N+1} e^{-x^3} R_N(x) \left(1 - \alpha\left(\frac{x}{\varepsilon}\right)\right)\right) \mathrm{d}x$$
$$= \int_{\mathbb{R}} e^{i\lambda x} D_\tau^M \left(x^{N+1} e^{-x^3} R_N(x) \left(1 - \alpha\left(\frac{x}{\varepsilon}\right)\right)\right) \mathrm{d}x$$

Similar to previous estimates of oscillation integrals, we have

$$|I_2| \le \int_{\mathbb{R}} \left| D_\tau^M \left(x^{N+1} e^{-x^3} R_N(x) \left(1 - \alpha\left(\frac{x}{\varepsilon}\right)\right)\right)\right| \mathrm{d}x \le C \int_{|x|>\varepsilon} \lambda^{-M} |x|^{N+1-3M} \,\mathrm{d}x \le C\lambda^{-M} \varepsilon^{N+2-3M}$$

where C only relies on M, N, ψ.

To summarize, we conclude that

$$\left| \int_{\mathbb{R}} e^{(i\lambda-1)x^3} x^{N+1} R_N(x)\,\mathrm{d}x \right| \le \varepsilon^{N+2} + \lambda^{-M}\varepsilon^{N+2-3M} = \lambda^{-\frac{N+2}{3}}$$

since we can let $\varepsilon = \lambda^{-\frac{1}{3}}$.

For general φ, we take Taylor expansion

$$\varphi(x) = \frac{\varphi'''(x_0)}{6}(x-x_0)^3 + O\left(|x-x_0|^4\right) = \frac{\varphi'''(x_0)}{6}(x-x_0)^3(1+\varepsilon(x))$$

where

$$\frac{1}{2} \le 1 + \varepsilon(x) \le \frac{3}{2}$$

for sufficiently small $|x - x_0|$.

Let

$$y = (x-x_0)(1+\varepsilon(x))^{\frac{1}{3}}$$

The function mapping x to y is a diffeomorphism when x locates in a small neighborhood of x_0, thus

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)}\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} e^{\frac{i\lambda\varphi'''(x_0)}{6}y^3} \left(\psi(x(y))\right) \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| \mathrm{d}y$$

Applying previous conclusion to φ, we obtain the final result

$$\int_{\mathbb{R}} e^{i\lambda\varphi(x)}\psi(x)\,\mathrm{d}x = \lambda^{-\frac{1}{3}} \sum_{j=1}^{N} a_j \lambda^{-\frac{j}{3}} + o\left(\lambda^{-\frac{N+2}{3}}\right)$$

□

10 Homework 10

10.1 Problem 10.1

If

$$\| \, (f\,\mathrm{d}\sigma)^\wedge \, \|_{L^{p'}(\mathbb{R}^n)} \lesssim \| \, f \, \|_{L^{q'}(\mathbb{S}^{n-1})}$$

holds for all $f \in L^{q'}(\mathbb{S}^{n-1})$, then

$$q \le \frac{n-1}{n+1} p'$$

Proof. We consider Knapp's example

$$C_\delta = \mathbb{S}^{n-1} \cap B_{\sqrt{2\delta}}(e_n)$$

and $f_\delta = \chi_{C_\delta}$.

Obviously, we have $f_\delta \in L^{q'}(\mathbb{S}^{n-1})$, so

$$\| (f_\delta \, \mathrm{d}\sigma)^\wedge \|_{L^{p'}(\mathbb{R}^n)} \leq C \| f_\delta \|_{L^{q'}(\mathbb{S}^{n-1})} = \left(\int_{\mathbb{S}^{n-1}} \chi_{C_\delta} \, \mathrm{d}\sigma \right)^{\frac{1}{q'}} = (\sigma(C_\delta))^{\frac{1}{q'}} \leq C\delta^{\frac{n-1}{q'}}$$

Abstract rectangle

$$R = \left(\prod_{j=1}^{n-1} \left[-c^{-1}\delta^{-1}, c^{-1}\delta^{-1} \right] \right) \times \left[-c^{-1}\delta^{-2}, c^{-1}\delta^{-2} \right]$$

for sufficiently large c, and we have

$$|(f_\delta \, \mathrm{d}\sigma)^\wedge| \geq \frac{1}{2}\sigma(C_\delta) = C\delta^{n-1}$$

thus

$$\| (f_\delta \, \mathrm{d}\sigma)^\wedge \|_{L^{p'}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |(f_\delta \, \mathrm{d}\sigma)^\wedge|^{p'} \right) \geq C\delta^{n-1}|R|^{\frac{1}{p'}} \geq C\delta^{n-1-\frac{n+1}{p'}}$$

To conclude, we have established an inequality

$$\delta^{n-1-\frac{n+1}{p'}} \leq C\delta^{\frac{n-1}{q'}} \Longrightarrow \delta^{n-1-\frac{n+1}{p'}-\frac{n-1}{q'}} \leq C$$

which is correct for small δ. As a result, we have

$$n - 1 - \frac{n+1}{p'} - \frac{n-1}{q'} \geq 0 \Longrightarrow q \leq \frac{n-1}{n+1}p'$$

□

10.2 Problem 10.2

Suppose that S is a bounded subset of a hyperplane in $\mathbb{R}^n$. Prove that if $\| \hat{f}|_S \|_{L^1(S)} \leq C \| f \|_{L^p(\mathbb{R}^n)}$ for all $f \in \mathcal{S}$, then necessarily $p = 1$. In other words, there cannot be a nontrivial restriction theorem for at (affine) surfaces.

Proof. With out loss of generality, we assume $S \subset \{x_n = 0\}$.

If a restriction theorem holds for some p, for an arbitrary t $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\| \hat{f}|_S \|_{L^1(S)} \leq C \| f \|_{L^p(\mathbb{R}^n)}$$

Let $f(x) = f(x', x_n)$, and we consider $f_\lambda(x', x_n) = f_\lambda(x', \lambda x_n)$. f_λ is still a Schwartz function, so the inequality holds. Direct computation shows

$$\hat{f}_\lambda(\xi) = \frac{1}{\lambda}\hat{f}\left(\xi', \frac{\xi}{\lambda}\right) \Longrightarrow \| \hat{f}_\lambda|_S \|_{L^1(S)} = \frac{1}{\lambda} \| \hat{f}|_S \|_{L^1(S)}$$

and

$$\| f \|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f_\lambda(x)|^p \right)^{\frac{1}{p}} = \left(\frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)|^p \right)^{\frac{1}{p}} = \frac{1}{\lambda^{\frac{1}{p}}} \| f \|_{L^p(\mathbb{R}^n)}$$

For $\forall \, \lambda > 0$, we have

$$\frac{1}{\lambda} \| \hat{f}|_S \|_{L^1(S)} \leq \frac{C}{\lambda^{\frac{1}{p}}} \| f \|_{L^p(\mathbb{R}^n)} \Longrightarrow \frac{1}{\lambda} \| \hat{f}|_S \|_{L^1(S)} \leq C\lambda^{1-\frac{1}{p}} \| f \|_{L^p(\mathbb{R}^n)}$$

That is to say, the inequality presented in restriction theorem does not always hold if $p \neq 1$. In fact, we can give a counterexample by selecting appropriate $\lambda > 0$. In conclusion, nontrivial restriction theorem for affine surfaces is incorrect.

□

11 Homework 11

11.1 Problem 11.1

Prove the following result:

Suppose π is a parallelogram in the (x, y) plane so that two of its sides lie on the lines $y = 0$ and $y = 1$, respectively. Then given any $\varepsilon > 0$, we can find parallelograms $\pi_1, \cdots, \pi_n$, each having two sides lying on the lines $y = 0$ and $y = 1$, with $\pi_i \subset \pi$, and

$$\left| \bigcup_{i=1}^{N} \pi_i \right| < \varepsilon$$

and so that any line segment in π that joins the lines $y = 0$ and $y = 1$ has a translate that is contained in one of the π_i.

Proof. As we constructed in the class, a triangle T can be cut off along its median. Then we can translate a part along the bottom base to obtain a new shape such that

$$|\Phi_h| = \alpha^2 |T| \qquad |\Phi_a| = 2(1 - \alpha^2)|T|$$

for some $\alpha \in (\frac{1}{2}, 1)$.

We could develop such process through dividing the original triangle into 2^n parts, every one of which has bottom base in identical length. Overlapping them in pairs, we obtain 2^{n-1} shapes mentioned above. Repeating such process n times, we obtain a final shape whose area satisfies

$$Area \leq \left(4^n + 2(1 - \alpha^2) \frac{1 - \alpha^{2n}}{1 - \alpha^2} \right) |T|$$

Therefore, the area tend to 0 as $\alpha \to 1^-$.

Back to our problem, we can view the parallelogram π as the combination of 2 triangles T_1 and T_2. In this case, $\forall \varepsilon > 0$, $\exists N$ sufficiently large, such that T_1 could be separated to 2^N components and these parts are identical in area. We can translate the components to construct a new shape S_1 such that $|S_1| < \frac{\varepsilon}{2}$.

Since T_1 nd T_2 are congruent, we could apply identical process to T_2 to obtain S_2 which is congruent to S_1. Therefore, we are able to translate S_2 globally to align each component with its counterpart in S_1, and thus obtain 2^N parallelograms $\pi_1, \cdots, \pi_{2^N}$ such that $\pi_i \subset \pi$.

As is proved in the class, these small parallelograms satisfy all properties in this problem.

□