第二十八讲 (2024.12.11)

Thm (Riesz-Fredholm)

设 $A \in \mathfrak{C}(X)$, $T \overset{\text{def}}{=} I - A$

(i) $\dim \ker(T) < \infty$

(ii) $\mathrm{Ran}(T) \overset{\text{closed}}{\hookrightarrow} X$ (闭值域算子)

(iii) T 单 $\Leftrightarrow$ T 满 (F. A.)

(iv) $\mathrm{Ran}(T) = \ker(T^*)^{\perp}$

这里对 $\mathcal{F} \subset X^*$

$$\mathcal{F}^{\perp} \overset{\text{def}}{=} \{x \in X : f(x) = 0, \ \forall f \in \mathcal{F}\}$$

称为 $\mathcal{F}$ 在 X 中的零化子. 同样, 对 $M \subset X$,

$$^{\perp}M \overset{\text{def}}{=} \{f \in X^* : f(x) = 0, \ \forall x \in M\}$$

称为 M 在 X^* 中的零化子.

Pf of (iv)

由下面 Lem 立得

Lem 设 $T \in \mathcal{L}(X)$.

(i) $\ker(T^*) = {}^{\perp}\mathrm{Ran}(T)$

(ii) $\ker(T^*)^{\perp} = \overline{\mathrm{Ran}(T)}$.

Pf (i)

$$f \in {}^{\perp}\mathrm{Ran}(T) \iff f(Tx) = 0, \quad \forall x \in X$$
$$\iff (T^{*}f)(x) = 0, \quad \forall x \in X$$
$$\iff T^{*}f = 0$$
$$\iff f \in \ker(T^{*})$$

(ii) 首先，由 (i)

$$\ker(T^{*})^{\perp} = ({}^{\perp}\mathrm{Ran}(T))^{\perp} \supset \mathrm{Ran}(T)$$

$\ker(T^{*})^{\perp}$ closed

$$\Longrightarrow \quad \overline{\mathrm{Ran}(T)} \subset \ker(T^{*})^{\perp}$$

Claim $\quad \ker(T^{*})^{\perp} \subset \overline{\mathrm{Ran}(T)}$

$\forall x \in \ker(T^{*})^{\perp}$

$$\Longrightarrow \quad x \in ({}^{\perp}\mathrm{Ran}(T))^{\perp}$$

回忆

$$x \in \overline{\mathrm{Ran}(T)} \iff \forall f \in X^{*} \text{ with } \underbrace{f(\mathrm{Ran}(T)) = \{0\}}_{\Updownarrow \; f \in {}^{\perp}\mathrm{Ran}(T)}$$

必有 $f(x) = 0$

而

$$\left.\begin{array}{l} x \in ({}^{\perp}\mathrm{Ran}(T))^{\perp} \\ f \in {}^{\perp}\mathrm{Ran}(T) \end{array}\right\} \Longrightarrow f(x) = 0$$

Thm (Riesz-Schauder)

设 $A \in \mathfrak{C}(X)$

(i) 如果 $\dim X = \infty$，则 $0 \in \sigma(A)$

(ii) $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$

（非零谱点一定是特征值）

(iii) 非零特征值的特征子空间是有限维的

(iv) 不同特征值的特征向量线性无关

(v) 0 是 $\sigma(A)$ 的唯一可能的极限点

Pf (i) 假设 $0 \in \rho(A)$

$$\Rightarrow A^{-1} \in \mathcal{L}(X)$$

$$\Rightarrow I = A^{-1}A \in \mathfrak{C}(X)$$

$$\Rightarrow \dim X < \infty$$

(ii) 即证明：$\forall \underbrace{\lambda \in \sigma_p(A)}_{\lambda I - A \text{ 单}}, \lambda \neq 0 \Rightarrow \lambda \in \rho(A)$

$$\xRightarrow{F.A.} \lambda I - A \text{ 是双射}$$

$$\xRightarrow{IMT} (\lambda I - A)^{-1} \in \mathcal{L}(X)$$

(iii) $\forall\, 0 \neq \lambda \in \sigma_p(A)$

$$\ker(\lambda I - A) = \ker\left(I - \frac{A}{\lambda}\right)$$

$$\overset{\text{Riesz-Fredholm}}{\Longrightarrow} \quad \dim \ker(\lambda I - A) < \infty$$

(iv) 略

(v) 假设 $\sigma(A)$ 有 ~~非~~ 极限点 $\lambda_0 \neq 0$

$\Longrightarrow \exists\, \lambda_n \in \sigma(A)$, $n = 1, 2, \cdots$ s.t.

$$\lambda_n \to \lambda_0$$

不妨设它们互不相同

$\lambda_n \to \lambda_0 \neq 0 \Longrightarrow$ 当 n 充分大时 $\lambda_n \neq 0$

不妨设 $\lambda_n \neq 0$. $\forall n$

$$\Longrightarrow \frac{1}{\lambda_n} \to \frac{1}{\lambda_0}$$

$$\Longrightarrow \sup_n \left|\frac{1}{\lambda_n}\right| < \infty$$

取 $x_n \in \ker(\lambda_n I - A)$, $n = 1, 2, \cdots$

$$\overset{\text{(iv)}}{\Longrightarrow} \{x_n\}_{n=1}^{\infty} \text{ 线性无关}$$

令

- $X_n \overset{def}{=} \text{span}\{x_1, \cdots, x_n\}$

$\Rightarrow X_n \overset{\text{closed}}{\hookrightarrow} X_{n+1}$

$\overset{\text{Riesz lem}}{\Rightarrow} \exists\, y_n \in X_n,\ \|y_n\| = 1$ s.t.

$$\text{dist}(y_n, X_{n-1}) > \frac{1}{2}$$

- 而 $y_n = \sum_{k=1}^{n} \alpha_k x_k$

$$\Rightarrow (\lambda_n I - A) y_n = \sum_{k=1}^{n} \alpha_k (\lambda_n I - A) x_k$$

$$= \sum_{k=1}^{n-1} \alpha_k (\lambda_n - \lambda_k) x_k \in X_{n-1}$$

$\forall n, m$, 设 $n > m$

- $\left\| A\left(\frac{y_n}{\lambda_n}\right) - A\left(\frac{y_m}{\lambda_m}\right) \right\|$

$$= \Big\| y_n - \Big[\underbrace{y_n - A\left(\frac{y_n}{\lambda_n}\right)}_{\in X_{n-1}} + \underbrace{A\left(\frac{y_m}{\lambda_m}\right)}_{\in X_m \subset X_{n-1}} \Big] \Big\|$$

$$> \text{dist}(y_n, X_{n-1}) > \frac{1}{2}$$

- $\Rightarrow \left\{ A\left(\frac{y_n}{\lambda_n}\right) \right\}_{n=1}^{\infty}$ 沒有收斂子列

另一方面

$$\sup_n \left|\frac{y_n}{\lambda_n}\right| < \infty$$

$\overset{A\ \text{cpt}}{\Longrightarrow}$ $\left\{A\left(\frac{y_n}{\lambda_n}\right)\right\}_{n=1}^{\infty}$ 有收敛子列，矛盾.

Cor $A \in \mathscr{C}(X) \Rightarrow \sigma(A)$ 至多可数

Pf: 令

$$E_k \overset{\text{def}}{=} \sigma_p(A) \cap \{\lambda \in \mathbb{C} : |\lambda| > \frac{1}{k}\}$$

$$\Rightarrow \sigma(A)\setminus\{0\} = \sigma_p(A)\setminus\{0\} \subset \bigcup_{k=1}^{\infty} E_k$$

claim $\#E_k < \infty$

假设不然 $\overset{B-W}{\Longrightarrow}$ E_k 有极限点 λ_0

而 $\mathrm{dist}(0, E_k) \geq \frac{1}{k}$

$\Rightarrow \lambda_0 \neq 0$

这与 $\sigma(A)$ 至多以 0 为极限点矛盾

Cor 如果 $\dim X = \infty$, $A \in \mathscr{C}(X)$

则只有以下三种可能：

Case 1 $\quad \sigma(A) = \{0\}$

Case 2 $\quad \sigma(A) = \{0, \lambda_1, \cdots, \lambda_n\}$

Case 3 $\quad \sigma(A) = \{0, \lambda_1, \lambda_2, \cdots\}$ with $\lambda_n \to 0$

其中 $\lambda_1, \lambda_2, \cdots \in \sigma_p(A)$

Pf: 令 $F_0 \stackrel{def}{=} \sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$

$$F_k \stackrel{def}{=} \sigma(A) \cap \left\{\lambda \in \mathbb{C} : \frac{1}{k+1} \leq |\lambda| < \frac{1}{k}\right\}$$

$\Rightarrow \sigma(A) \setminus \{0\} \subset \bigcup_{k=1}^{\infty} F_k$

且 $\# F_k < \infty$. $\forall k$ (问题)

按 F_k 顺次排列 $\lambda_1, \lambda_2, \cdots$

例: $A = 0 \Rightarrow \sigma(A) = \sigma_p(A) = \{0\}$

非平凡的例子

$A: \ell^2 \to \ell^2$

$(x_1, x_2, x_3, \cdots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$

$\Rightarrow A \in \mathfrak{C}(\ell^2)$ (HW)

$\sigma_p(A) = \phi. \quad \sigma(A) = \{0\}$

例：给定 $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{C}$，令

$$A_n : \ell^2 \to \ell^2$$

$$(x_1, x_2, \cdots) \mapsto (\lambda_1 x_1, \cdots, \lambda_n x_n, 0, \cdots)$$

$$\Rightarrow A_n \in \mathcal{B}(\ell^2) \qquad (\because A_n \in \mathcal{F}(\ell^2))$$

① $A_n e_k = \lambda_k e_k, \quad k = 1, \cdots, n$

$A_n e_{n+1} = 0$

$$\Rightarrow \{0, \lambda_1, \cdots, \lambda_n\} \subset \sigma_p(A)$$

另 $\forall \lambda \in \mathbb{C} \setminus \{0, \lambda_1, \cdots, \lambda_n\}$

$$(\lambda I - A_n)x = 0 \Leftrightarrow ((\lambda-\lambda_1)x_1, \cdots, (\lambda-\lambda_n)x_n, \lambda x_{n+1}, \cdots) = 0$$

$$\Leftrightarrow x = 0$$

$\Rightarrow$ $\lambda I - A_n$ 单

$\overset{\text{F.A.}}{\Rightarrow}$ $\lambda I - A_n$ 是双射

$\overset{\text{IMT}}{\Rightarrow}$ $\lambda \in \rho(A_n)$

$$\Rightarrow \sigma(A_n) = \sigma_p(A_n) = \{0, \lambda_1, \cdots, \lambda_n\}$$

例：设 $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}\setminus\{0\}$

with $\lambda_n \to 0$

$A: \ell^2 \to \ell^2$

$(x_1, x_2, \cdots) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \cdots)$

1° $\|Ax\|_2 \leq \left(\sup_n |\lambda_n|\right) \|x\|_2$

$\Rightarrow A \in \mathcal{L}(\ell^2)$

$\lambda_n \to 0 \Rightarrow \forall \varepsilon > 0, \exists N$ s.t.

$|\lambda_k| < \varepsilon, \quad \forall k \geq N$

$$\Rightarrow \|A - A_N\| = \sup_{\|x\|_2 = 1} \left(\sum_{k=N+1}^{\infty} |\lambda_k|^2 |x_k|^2\right)^{1/2}$$

$$\leq \varepsilon$$

$\Rightarrow \|A - A_N\| \to 0$ as $N \to \infty$

$A_N \in \mathcal{C}(\ell^2)$

$\Rightarrow A \in \mathcal{C}(\ell^2)$

2° $Ae_k = \lambda_k e_k$

$\Rightarrow \{\lambda_k\}_{k=1}^{\infty} \subset \sigma_p(A)$

且 $0 \notin \sigma_p(A)$

$\forall \lambda \in \mathbb{C}\setminus\{0, \lambda_1, \lambda_2, \cdots\}$,

$\Rightarrow \inf_k |\lambda - \lambda_k| > 0$

$\subseteq$

$$T : \ell^2 \to \ell^2$$

$$(x_1, x_2, \dots) \longmapsto \left(\frac{x_1}{\lambda - \lambda_1}, \frac{x_2}{\lambda - \lambda_2}, \dots \right)$$

$$\Rightarrow \quad \|Tx\|_2 \leq \left(\sup_k \frac{1}{|\lambda - \lambda_k|} \right) \|x\|_2$$

$$\Rightarrow \quad T \in \mathcal{L}(\ell^2)$$

u $\quad T = (\lambda I - A)^{-1}$

$$\Rightarrow \quad \lambda \in \rho(A)$$