第二十七讲（2024.12.9）

Riesz - Fredholm 理论

Thm 设 $A \in \mathfrak{C}(X)$，$T \overset{\text{def}}{=} I - A$

(i) $\dim \ker(T) < \infty$

(ii) $\mathrm{Ran}(T) \overset{\text{closed}}{\hookrightarrow} X$ （T 为闭值域算子）

(iii) T 单 $\iff$ T 满 （F.A. $\overset{\text{def}}{=}$ Fredholm Alternative）

(iv) $\mathrm{Ran}(T) = \mathrm{Ker}(T^*)^{\perp}$

回忆. 对 $\mathcal{F} \subset X^*$

$$\mathcal{F}^{\perp} \overset{\text{def}}{=} \{x \in X : f(x) = 0,\ \forall f \in \mathcal{F}\}$$

称为 $\mathcal{F}$ 在 X 中的零化子

(v) $\dim \ker(T) = \dim \ker(T^*)$

Rmk：为什么称 (iii) 为 F.A.（二择一律）？

二择一 $\nearrow$ $\forall y \in X$，$Tx = y$ 有唯一解（$\iff$ T 是双射）

二择一 $\searrow$ $Tx = 0$ 有非零解（$\iff$ T 不单 $\overset{\text{F.A}}{\iff}$ T 不满 $\iff$ $\exists y \in X$ s.t. $Tx = y$ 无解）

Thm $A \in \mathfrak{C}(X)$，$T \overset{\text{def}}{=} I - A \Rightarrow \dim \ker(T) < \infty$

Pf 令 $M \overset{\text{def}}{=} \ker(T)$

$S_M \overset{\text{def}}{=}$ M 中单位球面

$S_X \overset{\text{def}}{=}$ X 中 - — -

$$x \in S_M \iff \begin{cases} x \in S_X \\ (I - A)x = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x \in S_X \\ x = Ax \in A(S_X) \end{cases}$$

$$\Longrightarrow S_M \subset \underbrace{A(S_X)}_{\text{列紧}}$$

$$\Longrightarrow S_M \text{ 列紧}$$

$$\Longrightarrow \dim M < \infty$$

Thm $A \in \mathfrak{C}(X) \Longrightarrow \mathrm{Ran}(T)$ 闭

Pf 设 $\underbrace{\mathrm{Ran}(T) \ni y_n}_{\exists x_n \in X \text{ s.t. } y_n = Tx_n} \to y$

Case 1 : $\underbrace{\{x_n\}_{n=1}^{\infty} \text{ 有界}}$

$$\Downarrow A \text{ 紧}$$

$\{Ax_n\}_{n=1}^{\infty}$ 有收敛子列 $\{Ax_{n_k}\}_{k=1}^{\infty}$

设 $Ax_{n_k} \to u$

$$\xRightarrow{x_{n_k} = y_{n_k} + Ax_{n_k}} x_{n_k} \to y + u$$

$$\Longrightarrow y_{n_k} = Tx_{n_k} \to T(y+u)$$

$$\xRightarrow{y_{n_k} \to y} y = T(y+u) \in \mathrm{Ran}(T)$$

Case 2 $\{x_n\}_{n=1}^{\infty}$ 无界

令 $d_n \overset{\text{def}}{=} \mathrm{dist}(x_n, \underbrace{\mathrm{Ker}(T)}_{\text{有限维子空间}})$

$$\Longrightarrow \exists z_n \in \mathrm{Ker}(T) \text{ s.t.}$$

$$\|x_n - z_n\| = d_n .$$

Claim $\{x_n - z_n\}_{n=1}^{\infty}$ 有界.

假设不然, i.e. $\sup_n d_n = +\infty$

不妨设 $d_n \to +\infty$

令 $v_n \overset{\text{def}}{=} \dfrac{x_n - z_n}{\|x_n - z_n\|}$, $n=1,2,\cdots$

$\Longrightarrow \quad Tv_n = \dfrac{Tx_n - Tz_n}{d_n} = \dfrac{y_n}{d_n} \to 0$ ($\{y_n\}_{n=1}^{\infty}$ 有界 $d_n \to \infty$

另一方面,

$\|v_n\| = 1 \overset{A \text{ 紧}}{\Longrightarrow} \{Av_n\}_{n=1}^{\infty}$ 有收敛子列)

设 $Ax_{n_k} \to w$

$\overset{Tv_{n_k} \to 0}{\Longrightarrow} \quad v_{n_k} = Av_{n_k} + Tv_{n_k} \to w$

$\Longrightarrow \quad Tv_{n_k} \to Tw$

$\overset{Tv_{n_k} \to 0}{\Longrightarrow} \quad Tw = 0$

$\Longrightarrow \quad w \in \ker(T)$

而 $\forall z \in \ker(T)$

$$\|v_n - z\| = \frac{1}{d_n} \| x_n - (\underbrace{z_n + d_n z}_{\in \ker(T)}) \|$$

$$\geq \frac{d_n}{d_n} = 1$$

$\Longrightarrow \quad \mathrm{dist}(v_n, \mathrm{Ker}(T)) \geq 1, \ \forall n$

$\Longrightarrow \quad w \in \ker(T)$ 与 $v_{n_k} \to w$ 矛盾.

从而 $\{x_n - z_n\}_{n=1}^{\infty}$ 有界 且 $T(x_n - z_n) = Tx_n = y_n$

$\Longrightarrow$ 归化为 Case 1.

Thm (F.A.)

$A \in \mathcal{C}(X)$, $T \overset{def}{=} I - A$. 则 T 单 $\Longleftrightarrow$ T 满

Lem (i) $\ker(T) \subset \ker(T^2) \subset \cdots$

(ii) $\exists n$ s.t. $\ker(T^n) = \ker(T^{n+1})$

Pf (i) 平凡.

(ii) 假设 $\forall n$, $\underbrace{\ker(T^n)}_{\text{闭子空间}} \subsetneq \ker(T^{n+1})$

$\overset{\text{Riesz lem}}{\Longrightarrow}$ $\exists x_n \in \ker(T^{n+1})$, $\|x_n\| = 1$ s.t.

$\text{dist}(x_n, \ker(T^n)) > 1/2$

$\forall n, m$, 不妨设 $n > m$.

$$T^n(Tx_n + Ax_m) = \underbrace{T^{n+1}x_n}_{=0} + T^nAx_m = A(\underbrace{T^n x_m}_{=0}) = 0$$

$\Longrightarrow$ $Tx_n + Ax_m \in \ker(T^n)$

$\Longrightarrow$ $\|Ax_n - Ax_m\| = \|x_n - (Tx_n + Ax_m)\| > \frac{1}{2}$

$\Longrightarrow$ $\{Ax_n\}_{n=1}^{\infty}$ 没有收敛子列. 与 A 的紧性矛盾.

Pf of F.A.

1° "$\Longleftarrow$"

假设 T 满但不$\underbrace{\text{单}}_{\ker(T) \neq \{0\}}$

$\Longrightarrow$ $\exists\, 0 \neq x_0 \in \ker(T)$

$\overset{T\text{满}}{\Longrightarrow}$ $\exists\, x_1 \in X$ s.t. $Tx_1 = x_0$

$\exists\, x_2 \in X$ s.t. $Tx_2 = x_1$

$\vdots$

$\Rightarrow \quad 0 \neq x_0 = Tx_1 = T^2 x_2 = \cdots$

$\Rightarrow \quad \begin{cases} T^n x_n \neq 0 \\ T^{n+1} x_n = 0 \end{cases}$

$\Rightarrow \quad x_n \in \ker(T^{n+1}) \setminus \ker(T^n), \qquad n = 1, 2, \cdots$

与 Lem 矛盾

2° "$\Rightarrow$"

假设 T 单但不满

令 $X_1 \overset{def}{=} T(X) = \mathrm{Ran}(T)$

$\overset{T\text{不满}}{\Longrightarrow} \quad X_1 \underset{\text{闭子空间}}{\hookrightarrow} X, \quad X_1 \neq X$

$\overset{T\text{单, 闭值域}}{\Longrightarrow} \quad X_2 \overset{def}{=} T(X_1) \underset{\text{闭}}{\hookrightarrow} X_1.$

且 $X_2 \neq X_1$

[否则 $T(X_1) = X_1$, 取 $x_0 \in X \setminus X_1$
$\Rightarrow Tx_0 \in T(X) = X_1 = T(X_1)$
$\Rightarrow \exists x_0' \in X_1$ s.t. $Tx_0' = Tx_0$, 与 T 单矛盾]

$\vdots$

$X_n \overset{def}{=} T^n(X)$

$\vdots$

$\Rightarrow \quad X_{n+1} \overset{closed}{\hookrightarrow} X_n$ 且 $X_{n+1} \neq X_n$

$\overset{Riesz}{\Longrightarrow} \quad \exists x_n \in X_n, \ \|x_n\| = 1$ s.t.

$$\mathrm{dist}(x_n, X_{n+1}) > 1/2$$

$\forall n, m$, 设 $n > m$,

$$
\begin{aligned}
Ax_m - Ax_n &= -(x_m - Ax_m) + (x_n - Ax_n) + x_m - x_n \\
&= x_m - \underbrace{(x_n + Tx_n - Tx_n)}_{\in X_{m+1}}
\end{aligned}
$$

$\Longrightarrow \quad \|Ax_m - Ax_n\| \geq \operatorname{dist}(x_m, X_{m+1}) > \frac{1}{2}$

$\Longrightarrow \quad \{Ax_n\}_{n=1}^{\infty}$ 没有收敛子列，与 A 的紧性矛盾.

Thm $A \in \mathscr{C}(X)$. $T = I - A$，则

$$\operatorname{Ran}(T) = \operatorname{Ker}(T^*)^{\perp}$$

Pf 由下面 Lem 立得

Lem 设 $T \in \mathcal{L}(X)$，则

(i) $\operatorname{Ker}(T^*) = {}^{\perp}\operatorname{Ran}(T)$

(ii) $\operatorname{Ker}(T^*)^{\perp} = \overline{\operatorname{Ran}(T)}$

这里对 $M \subset X$, $\mathcal{F} \subset X^*$

${}^{\perp}M \overset{\text{def}}{=} \{f \in X^* : f(x) = 0, \forall f \in M\}$

$\mathcal{F}^{\perp} \overset{\text{def}}{=} \{x \in X : f(x) = 0, \forall f \in \mathcal{F}\}$

Pf (i)

$$
\begin{aligned}
f \in {}^{\perp}\operatorname{Ran}(T) &\iff f(Tx) = 0, \ \forall x \in X \\
&\iff (T^*f)(x) = 0, \ \forall x \in X \\
&\iff T^*f = 0 \\
&\iff f \in \operatorname{Ker}(T^*)
\end{aligned}
$$

(ii) 首先

$$\operatorname{Ker}(T^*)^{\perp} \overset{\text{by (i)}}{=} \left({}^{\perp}\operatorname{Ran}(T)\right)^{\perp} \supset \operatorname{Ran}(T)$$

$\Longrightarrow \quad \overline{\operatorname{Ran}(T)} \subset \operatorname{Ker}(T^*)^{\perp} \quad (\because \operatorname{Ker}(T^*)^{\perp}$ 闭$)$

再证明 $\operatorname{Ker}(T^*)^{\perp} \subset \overline{\operatorname{Ran}(T)}$

设 $x \in \ker(T^*)^{\perp}$

$\overset{\text{by (i)}}{\Longrightarrow} \quad x \in \left({}^{\perp}\mathrm{Ran}(T)\right)^{\perp}$

为证明 $x \in \overline{\mathrm{Ran}(T)}$,

$x \in \overline{\mathrm{Ran}(T)} \iff \forall f \in X^*$ with $\underbrace{f(\mathrm{Ran}(T)) = \{0\}}_{\Updownarrow \; f \in {}^{\perp}\mathrm{Ran}(T)}$

必有 $f(x) = 0$

而

$$\left.\begin{array}{l} x \in \left({}^{\perp}\mathrm{Ran}(T)\right)^{\perp} \\ f \in {}^{\perp}\mathrm{Ran}(T) \end{array}\right\} \Rightarrow f(x) = 0$$