第二十六讲 (2024.12.4.)

$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$

例：右移位算子 $A: \ell^2 \to \ell^2$

$(x_1, x_2, \cdots) \mapsto (0, x_1, x_2, \cdots)$

$\Rightarrow \sigma_p(A) = \phi, \quad \sigma_c(A) = \partial\mathbb{D}, \quad \sigma_r(A) = \mathbb{D}$

Pf $\|A\| = 1 \Rightarrow \sigma(A) \subset \overline{\mathbb{D}}$

1° $\sigma_p(A) = \phi$

否则 $\exists \lambda \in \mathbb{C}$, $\exists\, 0 \neq x \in \ell^2$ s.t.

$(0, x_1, x_2, \cdots) = Ax = \lambda x = (\lambda x_1, \lambda x_2, \cdots)$

$$\Rightarrow \begin{array}{l} \lambda x_1 = 0 \\ \lambda x_2 = x_1 \\ \lambda x_3 = x_2 \\ \vdots \end{array}$$

如果 $\lambda = 0$，则 $x = 0$

如果 $\lambda \neq 0$，则 $x_1 = 0 \Rightarrow x_2 = 0, \cdots \Rightarrow x = 0$ 矛盾

2° $\mathbb{D} \subset \sigma_r(A)$

设 $\lambda \in \mathbb{D}$，需证明：$\overline{\mathrm{Ran}(\lambda I - A)} \neq \ell^2$

$\Updownarrow$

$\mathrm{Ran}(\lambda I - A)^\perp \neq \{0\}$

令 $z = (1, \bar{\lambda}, \bar{\lambda}^2, \cdots)$

$\Rightarrow \langle (\lambda I - A)x, z \rangle$

$= \langle (\lambda x_1, \lambda x_2 - x_1, \lambda x_3 - x_2, \cdots), (1, \bar{\lambda}, \bar{\lambda}^2, \cdots) \rangle$

$= \lambda x_1 + \lambda^2 x_2 - \lambda x_1 + \lambda^3 x_3 - \lambda^2 x_2 + \cdots$

$= 0$

$\Rightarrow 0 \neq z \in \mathrm{Ran}(\lambda I - A)^\perp$

3° $\partial\mathbb{D} \subset \sigma_c(A)$.

设 $\lambda \in \partial\mathbb{D}$

Step 1 $\quad \mathrm{Ran}(\lambda I - A) \neq \ell^2$

$\mathrm{Ran}(\lambda I - A) \ni y = (\lambda I - A)x$

$$\Longrightarrow \begin{cases} y_1 = \lambda x_1 \\ y_k = \lambda x_k - x_{k-1}, & k \geq 2 \end{cases}$$

$$\Longrightarrow \begin{cases} y_1 = \lambda x_1 \\ \lambda^{k-1} y_k = \lambda^k x_k - \lambda^{k-1} x_{k-1}, & k \geq 2 \end{cases}$$

$$\Longrightarrow \sum_{k=1}^{n} \lambda^{k-1} y_k = \lambda^n x_n$$

假设 $\mathrm{Ran}(\lambda I - A) = \ell^2$

令 $y = e_1$

$\Longrightarrow \exists\, x \in \ell^2 \quad \text{s.t.} \quad e_1 = (\lambda I - A)x$

$\Longrightarrow \lambda^n x_n = 1, \quad n = 1, 2, \cdots$

$\Longrightarrow x = \left(\frac{1}{\lambda}, \frac{1}{\lambda^2}, \cdots\right)$ 与 $x \in \ell^2$ 矛盾

$(\because |\lambda| = 1)$

Step 2. $\quad \overline{\mathrm{Ran}(\lambda I - A)} = \ell^2$

只需证：$\mathrm{Ran}(\lambda I - A)^{\perp} = \{0\}$.

$\forall z \in \mathrm{Ran}(\lambda I - A)^{\perp}$

$$0 = \langle z, \underbrace{(\lambda I - A)e_n}_{(0, \cdots, 0, \lambda, -1, 0, \cdots)} \rangle = \bar{\lambda} z_n - z_{n+1}, \quad \forall n.$$

$\Longrightarrow z_{n+1} = \bar{\lambda} z_n, \quad n = 1, 2, \cdots$

$\overset{|\lambda| = 1}{\Longrightarrow} |z_{n+1}| = |z_n|, \quad \forall n$

$\overset{z \in \ell^2}{\Longrightarrow} z = 0$

4° $\quad \overline{\mathbb{D}} \subset \sigma_c(A) \cup \sigma_r(A) \subset \sigma(A) \subset \overline{\mathbb{D}}$

$\xRightarrow{2^\circ,\,3^\circ} \quad \sigma_c(A) = \partial\mathbb{D}, \quad \sigma_r(A) = \mathbb{D}$

紧算子

Def $\quad X, Y$ — Banach 空间

$A \in \mathcal{L}(X, Y)$

(i) 如果 A 把每个有界集映为列紧集，则称 A 紧，记为 $A \in \mathcal{C}(X, Y)$

(ii) 如果 A 把 X 中每个弱收敛序列映为 Y 中强收敛序列，则称 A 全连续

(iii) 如果 $\dim \mathrm{Ran}(A) < \infty$，则称 A 为有限秩算子，记为 $A \in \mathcal{F}(X, Y)$

Prop $\quad \mathcal{F}(X, Y) \hookrightarrow \mathcal{C}(X, Y)$

Pf $\quad$ 设 $A \in \mathcal{F}(X, Y)$

$$\forall M \overset{\text{bdd}}{\subset} X \Rightarrow A(M) \overset{\text{bdd}}{\subset} \underbrace{\mathrm{Ran}(A)}_{\text{有限维线性空间}}$$

$$\Rightarrow A(M) \text{ 列紧}$$

例: $\quad I \in \mathcal{C}(X) \iff \dim X < \infty$

例: $\quad$ 设 $K \in C([a,b]^2)$

$$(Tu)(s) \overset{\text{def}}{=} \int_a^b K(s,t)\, u(t)\, dt$$

则: $T: C[a,b] \to C[a,b]$ 紧

Pf: 设 $\mathcal{F} \overset{\text{bdd}}{\subset} C[a,b]$，要证明 $T(\mathcal{F})$ 列紧

记 $M \overset{def}{=} \sup\limits_{u\in\mathcal{F}} \|u\|$

$\Longrightarrow \|Tu\| \leq \|T\|\cdot M, \quad \forall u\in\mathcal{F}$

$\Longrightarrow T(\mathcal{F})$ 一致有界

Claim $T(\mathcal{F})$ 等度连续.

$\forall \varepsilon>0, \ \forall u\in\mathcal{F}$

$K(\cdot,\cdot)$ 一致连续 $\Longrightarrow \exists\, \delta>0$ s.t.

$$|K(s',t)-K(s'',t)| < \frac{\varepsilon}{M(b-a)}$$

$\forall s', s''\in[a,b]$ with $|s'-s''|<\delta$

$\forall t\in[a,b]$

$$\Longrightarrow |(Tu)(s')-(Tu)(s'')|$$

$$\leq \int_a^b |K(s',t)-K(s'',t)|\,|u(t)|\,dt$$

$< \varepsilon, \quad \forall s', s''\in[a,b]$ with $|s'-s''|<\delta$

$\forall u\in\mathcal{F}$

Prop $\mathcal{C}(X,Y) \overset{closed}{\hookrightarrow} \mathcal{L}(X,Y)$

Pf 设 $A_n\in\mathcal{C}(X,Y)$, $n=1,2,\cdots$ s.t. $\|A_n-A\|\to 0$

下证 A 紧

设 $M \overset{bdd}{\subset} X \quad \Longrightarrow \quad C \overset{def}{=} \sup\limits_{n} \|x\| < \infty$

Claim $A(M)$ 列紧

$\forall \varepsilon>0$, 取 N 充分大 s.t.

$$\|A_N - A\| < \frac{\varepsilon}{3C}.$$

$A_N(M)$ 列紧 $\Longrightarrow \exists\, x_1,\cdots,x_m\in M$ s.t.

$$A_N(M) \subset \bigcup_{k=1}^m B\left(A_N x_k, \frac{\varepsilon}{3}\right)$$

$\Longrightarrow$ $\forall x \in M$, $\exists k \in \{1, \cdots, m\}$ s.t.

$$\| A_N x - A_N x_k \| < \frac{\varepsilon}{3}$$

$\Longrightarrow$ $\| Ax - Ax_k \|$

$$\leq \underbrace{\| Ax - A_N x \|}_{\substack{\leq \|A - A_N\| C \\ < \varepsilon/3}} + \underbrace{\| A_N x - A_N x_k \|}_{< \varepsilon/3} + \underbrace{\| A_N x_k - A x_k \|}_{\substack{\leq \|A_N - A\| C \\ < \varepsilon/3}}$$

$< \varepsilon$

$\Longrightarrow$ $\{Ax_1, \cdots, Ax_m\}$ 是 $A(M)$ 的有穷 ε 网

Prop 紧算子值域可分

Pf $\quad \mathrm{Ran}(A) = \bigcup_{n=1}^{\infty} \underbrace{A(B(0,n))}_{\text{列紧} \Rightarrow \text{可分}}$

设 M_n 是 $A(B(0,n))$ 的可数稠密子集

$\Longrightarrow$ $\bigcup_{n=1}^{\infty} M_n$ 是 $\mathrm{Ran}(A)$ - - - - -

Prop. 紧算子与有界算子的复合是紧算子.

1° $\left.\begin{array}{l} A \in \mathfrak{C}(X, Y) \\ T \in \mathcal{L}(Y, Z) \end{array}\right\} \Longrightarrow T \circ A \in \mathfrak{C}(X, Z)$

2° $\left.\begin{array}{l} T \in \mathcal{L}(X, Y) \\ A \in \mathfrak{C}(Y, Z) \end{array}\right\} \Longrightarrow A \circ T \in \mathfrak{C}(X, Z)$

Pf 1° 设 $\{x_n\}_{n=1}^{\infty} \subset X$ 有界

$\xRightarrow{A\text{紧}}$ $\{Ax_n\}_{n=1}^{\infty}$ 有子列 $\{Ax_{n_k}\}_{k=1}^{\infty}$ 收敛

$\xRightarrow{T\text{有界}}$ $\{TAx_{n_k}\}_{k=1}^{\infty}$ 收敛.

2° 有界集 $M \xRightarrow{T\text{有界}} T(M)$ 有界 $\xRightarrow{A\text{紧}} A(T(M))$ 列紧

<u>Thm</u> 对 $A \in \mathcal{L}(X, Y)$

1° 紧 $\Rightarrow$ 全连续

2° 如果 X 自反，则 A 紧 $\Leftrightarrow$ A 全连续

Pf 1° 假设 A 紧但 $\underbrace{\text{不全连续}}$

$$\exists x_n \xrightarrow{w} x_0 \text{ 但 } \underbrace{\|Ax_n - Ax_0\| \not\to 0}$$

$\exists \varepsilon_0 > 0$，$\exists$ 子列 $\{x_{n_k}\}_{k=1}^{\infty}$

s.t.

$$\|Ax_{n_k} - Ax_0\| \geq \varepsilon_0$$

$x_{n_k} \xrightarrow{w} x_0 \xRightarrow{UBP} \{x_{n_k}\}_{k=1}^{\infty}$ 有界

$\xRightarrow{A\text{紧}} \{Ax_{n_k}\}_{k=1}^{\infty}$ 有收敛子列

不妨设 $Ax_{n_k} \to y$

另一方面，$\forall f \in Y^*$

$$f(Ax_{n_k} - Ax_0) = \underbrace{(A^* f)}_{\in X^*}(x_{n_k} - x_0) \to 0$$

$(\because x_{n_k} \xrightarrow{w} x_0)$

$\Rightarrow Ax_{n_k} \xrightarrow{w} Ax_0$

$\xRightarrow{Ax_{n_k} \to y} Ax_0 = y$

$\Rightarrow \|Ax_{n_k} - Ax_0\| \to 0$，矛盾

2° 设 $\{x_n\}_{k=1}^{\infty} \subset X$ 有界

$\xRightarrow[\text{Eberlein-S}]{X\text{自反}}$ 有子列 $x_{n_k} \xrightarrow{w} x_0$

$\xRightarrow{A\text{全连续}} \|Ax_{n_k} - Ax_0\| \to 0$