第二十五讲 (2024.12.2)

Def 算子值函数 $R_\lambda(A): \rho(A) \to \mathcal{L}(X)$

$$\lambda \mapsto (\lambda I - A)^{-1}$$

称为 A 的预解式 (resolvent)

Lem 设 $T \in \mathcal{L}(X)$, $\|T\| < 1$, 则

(i) $(I-T)^{-1} \in \mathcal{L}(X)$

(ii) $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$ (von-Neuman级数)

(iii) $\|(I-T)^{-1}\| \leqslant \dfrac{1}{1-\|T\|}$

Pf (i) 令

$$S_n \overset{def}{=} \sum_{k=0}^{n} T^k$$

$$\|S_{n+p} - S_n\| = \left\| \sum_{k=n+1}^{n+p} T^k \right\|$$

$$\leqslant \sum_{k=n+1}^{n+p} \|T\|^k < \frac{\|T\|^{n+1}}{1-\|T\|}$$

$\overset{\mathcal{L}(X)\text{完备}}{\Longrightarrow}$ $\exists S \in \mathcal{L}(X)$ s.t.

$$\|S_n - S\| \to 0$$

Claim $S = (I-T)^{-1}$

$$\|S_n(I-T) - I\| = \|I - T^{n+1} - I\|$$

$$\leqslant \|T\|^{n+1} \to 0 \quad \text{as } n \to \infty$$

$\Longrightarrow$ $\|S(I-T) - T\|$

$\leqslant \|S(I-T) - S_n(I-T)\| + \|S_n(I-T) - I\|$

$\leqslant \|S - S_n\| \, \|I - T\| + \|S_n(I-T) - I\|$

$\to 0$

$$\Rightarrow \quad S(I-T) = I$$

同理有 $(I-T)S = I$

(ii) $\|S_n - S\| \to 0 \quad \Rightarrow \quad (I-T)^{-1} = \sum_{n=0}^{\infty} T^n$

(iii) $\|S\| \leq \sup_n \|S_n\| \leq \dfrac{1}{1-\|T\|}$

<u>Thm</u> $\rho(A) \overset{\text{open}}{\subset} \mathbb{C}$

<u>Pf</u> 设 $\lambda_0 \in \rho(A)$

要证明 λ_0 是内点.

$$\lambda I - A = \lambda_0 I - A + (\lambda - \lambda_0) I$$

$$= (\lambda_0 I - A)\left[I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1} \right]$$

Lem $\Rightarrow$ 当 $|\lambda - \lambda_0| < \dfrac{1}{\|R_{\lambda_0}(A)\|}$ 时

$$B \overset{\text{def}}{=} \left[I + (\lambda - \lambda_0) R_{\lambda_0}(A) \right]^{-1} \in \mathcal{L}(X)$$

$$\Rightarrow \quad (\lambda I - A)^{-1} = B R_{\lambda_0}(A) \in \mathcal{L}(X)$$

$$\Rightarrow \quad \lambda \in \rho(A)$$

$$\Rightarrow \quad \mathbb{D}\left(\lambda_0, \frac{1}{\|R_{\lambda_0}(A)\|}\right) \subset \rho(A)$$

<u>Thm</u> $A \in \mathcal{L}(X) \quad \Rightarrow \quad \sigma(A) \subset \overline{\mathbb{D}(0, \|A\|)}$

<u>Pf</u> 即证明: $\mathbb{C} \setminus \overline{\mathbb{D}(0, \|A\|)} \subset \rho(A)$

$\Leftrightarrow \quad \forall \lambda \in \mathbb{C}$ with $|\lambda| > \|A\|$

$$(\lambda I - A)^{-1} \in \mathcal{L}(X)$$

而 $|\lambda| > \|A\| \Longrightarrow \left\|\frac{A}{\lambda}\right\| < 1$

$$\overset{\text{Lem}}{\Longrightarrow} \left(I - \frac{A}{\lambda}\right)^{-1} \in \mathcal{L}(X)$$

$$\Longleftrightarrow (\lambda I - A)^{-1} \in \mathcal{L}(X)$$

Cor $\sigma(A) \overset{\text{cpt}}{\subset} \mathbb{C}$

Def X — 复 Banach 空间

$\Omega \overset{\text{open}}{\subset} \mathbb{C}$

称算子值函数 $T: \Omega \to \mathcal{L}(X)$ 在 $\lambda_0 \in \Omega$ 全纯是指:

$\lambda \mapsto T_\lambda$

存在 λ_0 的邻域 U s.t. $\forall \lambda \in U$, $\exists S_\lambda \in \mathcal{L}(X)$ s.t.

$$\left\| \frac{T_{\lambda+z} - T_\lambda}{z} - S_\lambda \right\| \to 0 \quad \text{as} \quad |z| \to 0$$

Thm $\lambda \mapsto R_\lambda(A)$ 是 $\rho(A)$ 上算子值全纯函数.

Lem (Resolvent Identity)

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda) R_\lambda(A) R_\mu(A), \quad \forall \lambda, \mu \in \rho(A)$$

Pf
$$R_\lambda(A) = (\lambda I - A)^{-1} (\mu I - A)(\mu I - A)^{-1}$$
$$= (\lambda I - A)^{-1} \left[\lambda I - A + (\mu - \lambda) I \right] (\mu I - A)^{-1}$$
$$= (\mu I - A)^{-1} + (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$

Pf of Thm

Step 1 连续性

$\forall \lambda_0 \in \rho(A)$

$$\lambda I - A = (\lambda_0 I - A)\left[I + (\lambda - \lambda_0)(\lambda_0 I - A)^{-1} \right]$$

$\Rightarrow$ 当 $|\lambda - \lambda_0| < \dfrac{1}{\|R_{\lambda_0}(A)\|}$ 时

$$R_\lambda(A) = \left[I + (\lambda - \lambda_0) R_{\lambda_0}(A) \right]^{-1} R_{\lambda_0}(A)$$

$\Rightarrow$ 当 $|\lambda - \lambda_0| < \dfrac{1}{2\|R_{\lambda_0}(A)\|}$ 时

$$\|R_\lambda(A)\| \leq \left\| \left[I + (\lambda - \lambda_0) R_{\lambda_0}(A) \right]^{-1} \right\| \, \|R_{\lambda_0}(A)\|$$

$$\overset{\text{Lem}}{\leq} \frac{1}{1 - \frac{1}{2}} \|R_{\lambda_0}(A)\|$$

$$= 2\|R_{\lambda_0}(A)\| .$$

$\overset{\text{R.I.}}{\Longrightarrow}$ $\|R_\lambda(A) - R_{\lambda_0}(A)\|$

$$\leq |\lambda - \lambda_0| \, \|R_\lambda(A)\| \, \|R_{\lambda_0}(A)\|$$

$$\leq 2\|R_{\lambda_0}(A)\|^2 \, |\lambda - \lambda_0| \qquad \left(当 \ |\lambda - \lambda_0| < \frac{1}{2\|R_{\lambda_0}(A)\|} \right)$$

Step 2 全纯性

$$\left\| \frac{R_\lambda(A) - R_{\lambda_0}(A)}{\lambda - \lambda_0} + R_{\lambda_0}(A)^2 \right\|$$

$$\overset{\text{R.I.}}{=} \left\| -R_\lambda(A) R_{\lambda_0}(A) + R_{\lambda_0}(A)^2 \right\|$$

$$\leq \|R_{\lambda_0}(A)\| \, \|R_\lambda(A) - R_{\lambda_0}(A)\| \longrightarrow 0 \quad \text{as } \lambda \to \lambda_0$$

Thm (Gelfand , 谱不空定理)

$$0 \neq A \in \mathcal{L}(X) \implies \sigma(A) \neq \phi$$

Pf 假设 $\sigma(A) = \phi$

$\implies \rho(A) = \mathbb{C}$

$\implies \lambda \mapsto R_\lambda(A)$ 是算子值整函数.

$\implies \forall f \in \mathcal{L}(X)^*$,

$$u_f(\lambda) \overset{def}{=} f(R_\lambda(A)), \quad \lambda \in \mathbb{C}$$

是整函数.

$$\left[\because \left| \frac{u_f(\lambda) - u_f(\lambda_0)}{\lambda - \lambda_0} + f(R_{\lambda_0}(A)^2) \right| \leq \|f\| \left\| \frac{R_\lambda(A) - R_{\lambda_0}(A)}{\lambda - \lambda_0} + R_{\lambda_0}(A)^2 \right\| \to 0 \text{ as } \lambda \to \lambda_0 \right.$$

另一方面，当 $|\lambda| > 2\|A\|$ 时

$$\|R_\lambda(A)\| \leq \frac{1}{|\lambda|} \frac{1}{1 - \left\|\frac{A}{\lambda}\right\|} = \frac{1}{|\lambda| - \|A\|} \leq \frac{1}{\|A\|}$$

而 $\lambda \mapsto R_\lambda(A)$ 连续，在 $\overline{\mathbb{D}(0, 2\|A\|)}$ 上有界.

$\implies \exists C > 0$ s.t.

$$\|R_\lambda(A)\| \leq C, \quad \forall \lambda \in \mathbb{C}$$

$$\implies |u_f(\lambda)| \leq \|f\| \|R_\lambda(A)\| \leq C\|f\|, \quad \forall \lambda \in \mathbb{C}.$$

$\overset{\text{Liouville}}{\implies} u_f = \text{const}.$

$$\implies f(R_\lambda(A)) = f(R_\mu(A)), \quad \forall \lambda, \mu \in \mathbb{C}, \quad \forall f \in \mathcal{L}(X)^*$$

$\overset{HBT}{\Longrightarrow}$ $R_\lambda(A) = R_\mu(A)$. $\forall \lambda, \mu \in \mathbb{C}$

与 R.I. 矛盾.

<u>Def</u> 对 $A \in \mathcal{L}(X)$

$$r_\sigma(A) \overset{def}{=} \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

称为 A 的谱半径.

<u>Thm</u> (Gelfand, 谱半径公式)

$$r_\sigma(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

<u>Pf</u> <u>Step 1</u> $\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ 存在

令 $r \overset{def}{=} \inf_n \|A^n\|^{\frac{1}{n}}$

$\Longrightarrow \liminf_{n\to\infty} \|A^n\|^{\frac{1}{n}} \geq r$

另一方面, $\forall \varepsilon > 0$, $\exists m$ s.t.

$$\|A^m\|^{\frac{1}{m}} < r + \varepsilon$$

$\forall n \in \mathbb{N}$ 有唯一分解 $n = p_n m + q_n$ with $0 \leq q_n < m$

$$\Longrightarrow \|A^n\|^{\frac{1}{n}} \leq \|A^{p_n m}\|^{\frac{1}{n}} \|A^{q_n}\|^{\frac{1}{n}}$$

$$\leq \|A^m\|^{\frac{p_n}{n}} \|A\|^{\frac{q_n}{n}}$$

$$< (r+\varepsilon)^{\frac{p_n m}{n}} \|A\|^{\frac{q_n}{n}}$$

$\Longrightarrow \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r + \varepsilon$ $\left(\because \frac{q_n}{n} \to 0,\ \frac{p_n m}{n} \to 1 \right.$ as $n \to \infty$

$\Longrightarrow \limsup_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r$

Step 2 $\quad r_\sigma(A) \le \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$

幂级数 $\sum\limits_{n=0}^{\infty} \|A^n\| z^n$ 的收敛半径 $= \dfrac{1}{\lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}}$

$\overset{z=\frac{1}{\lambda}}{\Longrightarrow}$ 当 $|\lambda| > \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ 时

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{\lambda^{n+1}} \right\| < \infty \quad \text{(收敛圆内绝对收敛)}$$

$\overset{\mathcal{L}(X)\text{完备}}{\Longrightarrow}$ $\sum\limits_{n=0}^{\infty} \dfrac{A^n}{\lambda^{n+1}}$ 收敛

另一方面, $\left\| \left(\sum\limits_{n=1}^{N} \dfrac{A^n}{\lambda^{n+1}} \right) (\lambda I - A) - I \right\|$

$$= \left\| I - \frac{A^{N+1}}{\lambda^{N+1}} - I \right\| \to 0$$

$\Rightarrow \sum\limits_{n=1}^{\infty} \dfrac{A^n}{\lambda^{n+1}} = (\lambda I - A)^{-1} = R_\lambda(A)$

$\Rightarrow R_\lambda(A) \in \mathcal{L}(X)$

$\Rightarrow \lambda \in \rho(A)$

$\Longrightarrow r_\sigma(A) \le \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$

Step 3 $\quad r_\sigma(A) \ge \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$

设 $|\lambda| > r_\sigma(A)$

$\Rightarrow \lambda \in \rho(A)$

$\Rightarrow \forall f \in \mathcal{L}(X)^*$, $f(R_\lambda(A))$ 在 λ 全纯

$\Rightarrow f(R_\lambda(A))$ 在圆环 $|\lambda| > r_\sigma(A)$ 内全纯

故可展为收敛的 Laurent 级数.

另一方面，由 Step 2，当 $|\lambda| > \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$ 时

$$R_\lambda(A) = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$$

$$\Longrightarrow \quad f(R_\lambda(A)) = \sum_{n=0}^{\infty} \frac{f(A^n)}{\lambda^{n+1}}$$

Laurent 展式唯一 $\Longrightarrow$ 这一展式在 $|\lambda| > r_\sigma(A)$ 上也成立.

在内部绝对收敛 $\Longrightarrow$ $\forall \varepsilon > 0$

$$\sum_{n=0}^{\infty} \frac{|f(A^n)|}{(r_\sigma(A)+\varepsilon)^{n+1}} < \infty$$

记

$$T_n \overset{\text{def}}{=} \frac{A^n}{(r_\sigma(A)+\varepsilon)^{n+1}}$$

收敛级数通项有界 $\Longrightarrow$ $\sup\limits_n |f(T_n)| < \infty, \quad \forall f \in \mathcal{L}(X)^*$

$\overset{\text{UBP}}{\Longrightarrow}$ $C \overset{\text{def}}{=} \sup\limits_n \|T_n\| < \infty$

$$\Longrightarrow \quad \|A^n\| \leq C\,(r_\sigma(A)+\varepsilon)^{n+1}$$

$$\Longrightarrow \quad \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r_\sigma(A) + \varepsilon$$

$$\Longrightarrow \quad \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} \leq r_\sigma(A)$$

HW: Ex. 2.6.1
2.6.2