第二十三讲 (2024.11.25)

<u>Thm</u> (Pettis) 自反空间的闭子空间自反.

<u>Pf</u> 设 X 自反. $Y \overset{闭}{\hookrightarrow} X$

只需证: $\forall \mathfrak{z} \in Y^{**}$, $\exists y \in Y$ s.t.

$$\mathfrak{z}(f) = f(y), \quad \forall f \in Y^*$$

$$(y = i^{-1}(\mathfrak{z}), \quad \mathfrak{z} = y^{**})$$

定义映射 $T: X^* \to Y^*$

$$f \mapsto f|_Y$$

$\Rightarrow T \in \mathcal{L}(X^*, Y^*)$

$\Rightarrow T^* \in \mathcal{L}(Y^{**}, X^{**})$ s.t.

$$(T^*\mathfrak{z})(f) = \mathfrak{z}(Tf), \quad \forall f \in X^*$$

X 自反 $\overset{def}{\Longleftrightarrow}$ 自然映射 $i: X \to X^{**}$ 是满射

$\overset{T^*\mathfrak{z} \in X^{**}}{\Longrightarrow}$ $\exists y \in X$ s.t. $T^*\mathfrak{z} = y^{**}$

$\Rightarrow (T^*\mathfrak{z})(f) = y^{**}(f) = f(y), \quad \forall f \in X^*$ (*)

<u>Claim 1</u> $y \in Y$

假设不然, 则 $\exists \tilde{f} \in X^*$ s.t.

$$\underbrace{\tilde{f}(Y) = \{0\}}_{\Downarrow} \quad 而 \quad \tilde{f}(y) = dist(y, Y) > 0$$

$T\tilde{f} = \tilde{f}|_Y = 0 \Rightarrow \tilde{f}(y) = (T^*\mathfrak{z})(\tilde{f}) = \mathfrak{z}(T\tilde{f}) = 0$

矛盾

<u>Claim 2</u> $\xi(f) = f(y), \quad \forall f \in Y^*$

$\forall f \in Y^*, \ \exists F \in X^* \quad \text{s.t.} \quad f = TF \quad (\text{by HBT})$

$$\Rightarrow \quad \xi(f) = \xi(TF) = (T^*\xi)(F) \underset{\text{由}(*)}{=} F(y) \underset{y\in Y \text{ 且 } F|_Y = f}{=} f(y)$$

<u>Def</u> $(X, \|\cdot\|)$

称 $\{x_n\}_{n=1}^{\infty} \subset X$ 弱收敛于 $x_0 \in X$ 是指:

$$f(x_n) \to f(x_0), \quad \forall f \in X^*$$

记为 $x_n \xrightarrow{w} x_0$ 或 $x_n \rightharpoonup x_0$

x_0 称为 $\{x_n\}_{n=1}^{\infty}$ 的弱极限.

<u>Prop</u> 弱极限(如果存在)唯一.

<u>Pf</u> 设 $\begin{cases} x_n \xrightarrow{w} x_0 \\ x_n \xrightarrow{w} y_0 \end{cases}$

$$\Rightarrow \begin{cases} f(x_n) \to f(x_0), \\ f(x_n) \to f(y_0), \end{cases} \quad \forall f \in X^*$$

$$\Rightarrow \quad f(x_0) = f(y_0), \quad \forall f \in X^*$$

$$\overset{\text{HBT}}{\Rightarrow} \quad x_0 = y_0$$

<u>Def</u> 范数拓扑下的收敛称为强收敛.

<u>Prop</u> 强收敛 $\Rightarrow$ 弱收敛.

<u>Pf</u> $\|x_n - x_0\| \to 0$

$$\Rightarrow \quad |f(x_n) - f(x_0)| \le \|f\| \, \|x_n - x_0\| \to 0, \quad \forall f \in X^*$$

例： $L^2(\mathbb{T})$ 中

$$e_k(t) \overset{\text{def}}{=} e^{-2\pi i k t}, \quad t \in [-\tfrac{1}{2}, \tfrac{1}{2}), \quad k \in \mathbb{Z}.$$

$$\Rightarrow \quad e_k \xrightarrow{w} 0 \quad \text{as } |k| \to \infty$$

$\forall\ f \in (L^2(\mathbb{T}))^*$, $\exists!\ v \in L^2(\mathbb{T})$ s.t.

$$f(u) = \int_{-1/2}^{1/2} u(t) v(t)\, dt, \quad u \in L^2(\mathbb{T})$$

$$\Rightarrow \quad f(e_k) = \int_{-1/2}^{1/2} v(t)\, e^{-2\pi i k t}\, dt = \hat{v}(k) \longrightarrow 0$$

as $|k| \to \infty$

(Riemann-Lebesgue lem)

Thm $\dim X < \infty \Rightarrow$ 弱收敛与强收敛等价.

Pf 设 $\dim X = m$ 则 $\{e_1, \cdots, e_m\}$ 是 X 的一个基.

$$\overset{\text{Ex. 2.4.7}}{\Longrightarrow} \quad \exists\ f_1, \cdots, f_m \in X^* \quad (\text{对偶基}) \quad \text{s.t.}$$

$$f_j(e_k) = \delta_{jk}, \quad j, k = 1, 2, \cdots, m$$

设 $\underset{\sum_{k=1}^m \alpha_k^{(n)} e_k}{\underset{\parallel}{x_n}} \xrightarrow{w} \underset{\sum_{k=1}^m \alpha_k^{(0)} e_k}{\underset{\parallel}{x_0}}$

$$\Rightarrow \quad \underbrace{f_j(x_n)}_{\alpha_j^{(n)}} \to \underbrace{f_j(x_0)}_{\alpha_j^{(0)}}, \quad j = 1, \cdots, m$$

$$\Rightarrow \quad \|x_n - x_0\|_\infty \to 0 \quad \text{with } \|x\|_\infty \overset{\text{def}}{=} \max_{1 \le k \le m} |\alpha_k|$$

有限维空间范数等价

$$\Rightarrow \quad \|x_n - x_0\| \to 0$$

(Rmk: 逆命题不成立
反例: Schur 空间)

Thm (Mazur)

$$x_n \xrightarrow{w} x_0 \quad \Rightarrow \quad x_0 \in \overline{\operatorname{conv}(\{x_n\}_{n=1}^\infty)}$$

Pf：令 $C \stackrel{def}{=} \overline{\text{conv}(\{x_n\}_{n=1}^{\infty})}$

假设 $x_0 \notin C$

$\stackrel{\text{Ascoli}}{\Longrightarrow}$ $\exists f \in X^*$，$\exists \alpha \in \mathbb{R}$ s.t.

$$\sup_{x \in C} f(x) < \alpha < f(x_0)$$

$\Longrightarrow$ $f(x_n) < \alpha < f(x_0)$，$n = 1, 2, \cdots$

与 $f(x_n) \to f(x_0)$ 矛盾

Def 称 $\{f_n\}_{n=1}^{\infty} \subset X^*$ 弱*收敛于 $f \in X^*$ 是指：

$$f_n(x) \to f(x), \quad \forall x \in X$$

记为 $f_n \xrightarrow{w^*} f$

Rmk：X^* 中

强收敛 $\Rightarrow$ 弱收敛 $\Rightarrow$ 弱*收敛.

$$\begin{aligned} f_n \xrightarrow{w} f &\stackrel{def}{\Longleftrightarrow} \Lambda(f_n) \to \Lambda(f), \quad \forall \Lambda \in X^{**} \\ &\Longrightarrow x^{**}(f_n) \to x^{**}(f), \quad \forall x \in X \\ &\Longleftrightarrow f_n(x) \to f(x), \quad \forall x \in X \\ &\stackrel{def}{\Longleftrightarrow} f_n \xrightarrow{w^*} f \end{aligned}$$

Prop X 自反 $\Rightarrow$ X^* 中弱*收敛与弱收敛等价

Rmk：逆命题不成立，反例：ℓ^{∞}

Thm $(X, \|\cdot\|)$

$$x_n \xrightarrow{w} x_0 \Longleftrightarrow \begin{cases} \sup_n \|x_n\| < \infty \quad \text{（从而弱收敛序列有界）} \\ \exists \mathcal{F} \stackrel{dense}{\subset} X^* \text{ s.t.} \\ f(x_n) \to f(x_0), \quad \forall f \in \mathcal{F} \end{cases}$$

Pf $\quad x_n \xrightarrow{w} x_0$

$\iff \quad f(x_n) \to f(x_0), \quad \forall f \in X^*$

$\iff \quad x_n^{**}(f) \to x_0^{**}(f), \quad \forall f \in X^*$

$$\overset{BS}{\iff} \quad \begin{cases} \sup\limits_n \|x_n^{**}\| < \infty \\ \exists\, \mathcal{F} \overset{\text{dense}}{\subset} X^* \quad \text{s.t.} \quad x_n^{**}(f) \to x_0^{**}(f) \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall f \in \mathcal{F} \end{cases}$$

Thm $\quad X$ — Banach 空间

$$f_n \xrightarrow{w^*} f \quad \iff \quad \begin{cases} \sup\limits_n \|f_n\| < \infty \\ \exists\, M \overset{\text{dense}}{\subset} X \quad \text{s.t.} \\ f_n(x) \to f(x), \quad \forall x \in M. \end{cases}$$

Def $\quad (X, \|\cdot\|)$

(1) 称 $M \subset X$ 弱列紧 是指 M 中任一序列都有弱收敛子列.

(2) 称 $\mathcal{F} \subset X^*$ 弱*列紧 是指 $\mathcal{F}$ 中任一序列都有弱*收敛子列

Thm（可分 Banach–Alaoglu Thm）

$\qquad X$ 可分 $\Rightarrow$ X^* 中有界集弱*列紧

Pf $\quad$ 设 $\{f_n\}_{n=1}^{\infty} \subset X^*$ 有界

$$c \overset{\text{def}}{=} \sup_n \|f_n\|$$

X 可分 $\Rightarrow$ $\exists\, \{x_n\}_{n=1}^{\infty} \overset{\text{dense}}{\subset} X$

$\forall m$, $\{f_n(x_m)\}_{n=1}^{\infty}$ 是有界数列. 故有收敛子列

$$\left(\quad |f_n(x_m)| \leq C\|x_m\| \quad \right)$$

对角线法

$$\Longrightarrow \{f_n\}_{n=1}^{\infty} \text{ 有子列 } \{f_{n_k}\}_{k=1}^{\infty} \quad s.t.$$

$$\forall m. \quad \{f_{n_k}(x_m)\}_{k=1}^{\infty} \text{ 收敛}$$

<u>Claim</u> $\exists f \in X^* \quad s.t. \quad f_{n_k} \xrightarrow{w^*} f$

$$\forall x \in X, \quad \forall \varepsilon > 0, \quad \exists x_m \in \{x_n\}_{n=1}^{\infty} \quad s.t.$$

$$\|x - x_m\| < \frac{\varepsilon}{3C}$$

$$\Longrightarrow |f_{n_{k+p}}(x) - f_{n_k}(x)|$$

$$\leq \underbrace{|f_{n_{k+p}}(x) - f_{n_{k+p}}(x_m)|}_{\substack{\leq C\|x - x_m\| \\ < \varepsilon/3}} + \underbrace{|f_{n_{k+p}}(x_m) - f_{n_k}(x_m)|}_{\substack{< \varepsilon/3 \\ \text{当} k \text{充分大}, \forall p}} + \underbrace{|f_{n_k}(x_m) - f_{n_k}(x)|}_{\substack{\leq C\|x - x_m\| \\ < \varepsilon/3}}$$

$$< \varepsilon, \quad \text{当 } k \text{ 充分大}, \ \forall p$$

$$\Longrightarrow f(x) \stackrel{def}{=} \lim_{k\to\infty} f_{n_k}(x) \text{ 存在}.$$

$$\underline{\text{且}} \quad |f(x)| \leq \sup_n |f_n(x)| \leq \sup_n \|f_n\| \, \|x\|$$

$$\Longrightarrow f \in X^* \quad \underline{\text{且}} \quad f_{n_k} \xrightarrow{w^*} f$$

<u>Thm</u> (Alaoglu)

$(X, \|\cdot\|)$

X^*中闭单位球弱$*$紧

<u>Thm</u> (Eberlein-Smulian)

X — 自反空间

(1) X中有界集弱列紧

(2) X中闭单位球弱自列紧