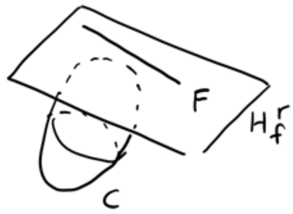
第二十讲 (2024.11.13)



Cor (Mazur)

X — 实赋范空间

C — 开凸集

F — 线性子流形 (i.e. 子空间的平移)

$$C \cap F = \emptyset \implies \exists\, H_f^r \text{ 闭 } s.t. \begin{cases} F \subset H_f^r \\ \sup\limits_{x \in C} f(x) \le r \end{cases}$$

Pf $F = M + x_0$ with $M \hookrightarrow X$

$$\overset{HST1}{\Longrightarrow} \exists f \in X^*, \ \exists s \in \mathbb{R} \quad s.t.$$

$$\sup_{x \in C} f(x) \le s \le \inf_{y \in F} f(y) = \inf_{z \in M} f(z) + f(x_0)$$

$$\Longrightarrow \inf_{z \in M} f(z) \ge s - f(x_0)$$

$$\Longrightarrow f|_M = 0$$

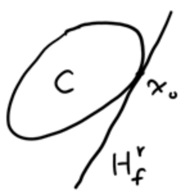
$$\Longrightarrow M \subset H_f^0$$

$$\Longrightarrow F \subset H_f^r \quad \text{with } r = f(x_0)$$

同时

$$\sup_{x \in C} f(x) \le s \le f(x_0) = r$$

Def 称超平面 H_f^r 为凸集 C 在 x_0 处的支撑超平面 (supporting hyperplane) 是指:



(i) C 完全落在 H_f^r 的一侧,

(ii) $x_0 \in \overline{C} \cap H_f^r$

即 $\sup\limits_{x \in C} f(x) \le r = f(x_0)$ 或 $\inf\limits_{x \in C} f(x) \ge r = f(x_0)$.

Thm X — 实赋范空间

C — 有内点的闭凸集

$\forall x_0 \in \partial C$ 处均有 C 的一个支撑超平面.

Pf 令

$$E \overset{def}{=} \mathrm{int}(C) \quad \text{开凸集}$$

$$F = \{x_0\}$$

$$\overset{\text{Mazur}}{\Longrightarrow} \quad \exists f \in X^*, \ \exists r \in \mathbb{R} \ \text{ s.t.}$$

$$\sup_{x \in E} f(x) \le r \quad \text{且} \quad \{x_0\} \subset H_f^r$$

$$\overset{f \text{ 连续}}{\Longrightarrow} \quad \sup_{x \in C} f(x) \le r = f(x_0)$$

例: $C = B(0, r)$

$\forall x_0 \in \partial B(0, r)$ 处均有 C 的支撑超平面.

Pf $\exists f \in X^*$, $\|f\| = 1$ s.t. $f(x_0) = \|x_0\| = r$

而

$$\sup_{x \in C} f(x) \le \|f\| \sup_{x \in C} \|x\| = r$$

对偶空间 (dual space)

$$X^* \overset{def}{=} \mathcal{L}(X, \mathbb{K})$$

例: H — Hilbert 空间

$$H^* = H$$

即 $J: H \to H^*$ 是线性等距同构 (赋范空间之间)

$$y \mapsto f_y$$

$$f_y(x) = \langle x, y \rangle$$

Thm (Riesz) 设 $(\Omega, \mathfrak{M}, \mu)$ 是 σ-有限测度空间

设 $1 \le p < \infty$, $p' \overset{\text{def}}{=} \begin{cases} \frac{p}{p-1}, & \text{if } 1 < p < \infty \\ \infty & \text{if } p = 1 \end{cases}$

$$(L^p)^* = L^{p'}$$

即

(i) $\forall g \in L^{p'}$, $\Lambda_g(f) \overset{\text{def}}{=} \int f g \, d\mu$

则 $\Lambda_g \in (L^p)^*$ 且 $\|\Lambda_g\| = \|g\|_{p'}$

(ii) $\forall \Lambda \in (L^p)^*$, $\exists !\, g \in L^{p'}$ s.t. $\Lambda = \Lambda_g$

故

$$J: \begin{aligned} L^{p'} &\to (L^p)^* \\ g &\mapsto \Lambda_g \end{aligned}$$ 是线性等距同构.

Pf of (i)

Case 1 $\quad 1 < p < \infty$

$\forall f \in L^p$

$$|\Lambda_g(f)| = \left| \int f g \right| \overset{\text{Hölder}}{\le} \|g\|_{p'} \|f\|_p$$

$\Rightarrow \Lambda_g \in (L^p)^*$ 且 $\|\Lambda_g\| \le \|g\|_{p'}$

为证明 $\|\Lambda_g\| \ge \|g\|_{p'}$, 令

$$\tilde{f} \overset{\text{def}}{=} |g|^{p'-1} \operatorname{sgn}(g)$$

$$\Rightarrow \begin{cases} |\tilde{f}|^p = |g|^{(p'-1)p} = |g|^{p'} \\ \tilde{f} g = |g|^{p'} \end{cases} \qquad \left(p = \frac{p'}{p'-1} \right)$$

$$\Rightarrow \begin{cases} \|\tilde{f}\|_p^p = \|g\|_{p'}^{p'} \\ \Lambda_g(\tilde{f}) = \|g\|_{p'}^{p'} \end{cases}$$

$$\Rightarrow \quad \frac{|\Lambda_g(\tilde{f})|}{\|\tilde{f}\|_p} = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'/p}} = \|g\|_{p'}^{p'(1-\frac{1}{p})} = \|g\|_{p'}$$

$$\Rightarrow \quad \|\Lambda_g\| \geq \|g\|_{p'}$$

<u>Case 2</u> $\quad p = 1$

<u>Step 1</u> 先假设 μ 是有限测度.

$$|\Lambda_g(f)| = \left| \int f g \, d\mu \right| \leq \|g\|_\infty \|f\|_1$$

$$\Rightarrow \quad \Lambda_g \in (L^1)^* \quad \text{即} \quad \|\Lambda_g\| \leq \|g\|_\infty$$

<u>Claim</u> $\quad \|\Lambda_g\| \geq \|g\|_\infty$

令

$$E_k \overset{def}{=} \left\{ x \in \Omega : |g(x)| > \|\Lambda_g\| + \frac{1}{k} \right\}$$

$$f_k \overset{def}{=} \chi_{E_k} \cdot \operatorname{sgn}(g)$$

$k = 1, 2, \cdots$

$$\Rightarrow \quad \|f_k\|_1 = \int_{E_k} |\operatorname{sgn}(g)| \, d\mu \leq \mu(E_k)$$

$$\begin{aligned}
\Rightarrow \quad \|\Lambda_g\| \, \mu(E_k) &\geq \|\Lambda_g\| \, \|f_k\|_1 \\
&\geq |\Lambda_g(f_k)| \\
&= \left| \int \chi_{E_k} \cdot \operatorname{sgn}(g) \cdot g \, d\mu \right| \\
&= \int_{E_k} |g| \, d\mu \\
&\geq \left(\|\Lambda_g\| + \frac{1}{k} \right) \mu(E_k)
\end{aligned}$$

$\mu(E_k) \leq \mu(\Omega) < \infty$

$$\Rightarrow \quad \mu(E_k) = 0$$

$$\Rightarrow \quad \{ x \in \Omega : |g(x)| > \|\Lambda_g\| \} = \bigcup_{k=1}^{\infty} E_k \text{ 是零集}$$

$$\Rightarrow \quad \|g\|_\infty \leq \|\Lambda_g\|$$

Step 2 $\quad \Omega = \bigcup_{n=1}^{\infty} \Omega_n \quad$ with $\mu(\Omega_n) < \infty$

$$E_{k,n} \overset{\text{def}}{=} E_k \cap \Omega_n$$

$$\overset{\text{Step 1}}{\Longrightarrow} \quad \mu(E_{k,n}) = 0$$

$$\Longrightarrow \quad E_k \subset \bigcup_{n=1}^{\infty} E_{k,n} \text{ 为零集}$$

为证明 (ii)，需要以下 Lem

Lem 设 $g \in L^1$，~~如~~ 若 $\exists\, C > 0$ s.t.

$$\left| \int f g \right| \le C \|f\|_p, \quad \forall f \in L^\infty$$

则 $g \in L^{p'}$ 且 $\|g\|_{p'} \le C$

Pf $\quad$ Case 1 $\quad 1 < p < \infty$，$\mu(\Omega) < \infty$

令

$$g_n \overset{\text{def}}{=} g \cdot \chi_{\{|g| \le n\}}$$

$$f_n \overset{\text{def}}{=} |g_n|^{p'-1} \operatorname{sgn}(g_n)$$

$$\Longrightarrow \begin{cases} f_n \in L^\infty \\ \|f_n\|_p^p = \int |g_n|^{(p'-1)p} \, d\mu = \|g_n\|_{p'}^{p'} \\ f_n g = f_n g_n = |g_n|^{p'} \end{cases}$$

$$\Longrightarrow \quad \|g_n\|_{p'}^{p'} = \left| \int f_n g \right|$$

$$\le C \|f_n\|_p \quad \text{（由假设）}$$

$$= C \|g_n\|_{p'}^{p'/p}$$

$$\Longrightarrow \quad \|g_n\|_{p'} \le C$$

$$\Rightarrow \quad \int |g|^{p'} d\mu = \int \lim_{n\to\infty} |g_n|^{p'} d\mu$$

$$\overset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int |g_n|^{p'} d\mu$$

$$\leq C^{p'}$$

$$\Rightarrow \quad g \in L^{p'} \ \underline{且} \ \|g\|_{p'} \leq C$$

<u>Case 2</u> $\quad p = 1$

令 $\quad E_k = \{ |g| > C + \frac{1}{k} \}$

$\quad f_k = \chi_{E_k} \operatorname{sgn}(g)$

(HW)

<u>HW</u> : Ex. 2.4.13 . 2.4.14