第十八讲 (2024.11.6)

Thm (HBT)

$(X, \|\cdot\|)$

$M \hookrightarrow X$

$\forall f \in M^*$, $\exists F \in X^*$ s.t.

$F|_M = f$ 且 $\|F\| = \|f\|$.

Cor 1 $\forall x_0 \in X$, $\exists f \in X^*$ with $\|f\| = 1$ s.t.

$f(x_0) = \|x_0\|$.

Pf 令 $M \overset{def}{=} span\{x_0\}$

$f_0 : M \to \mathbb{K}$

$\lambda x_0 \mapsto \lambda \|x_0\|$

$\Rightarrow |f_0(x)| = |\lambda| \|x_0\| = \|x\|$, $\forall x \in M$

$\Rightarrow f_0 \in M^*$ 且 $\|f_0\| = 1$.

$\overset{HBT}{\Rightarrow} \exists f \in X^*$ s.t.

$$\begin{cases} f|_M = f_0 & \Rightarrow f(x_0) = f_0(x_0) = \|x_0\| \\ \|f\| = \|f_0\| = 1 \end{cases}$$

Cor 2 $X \neq \{0\} \Rightarrow X^* \neq \{0\}$

Pf $\exists\, 0 \neq x_0 \in X$

$\overset{Cor\,1}{\Rightarrow} \exists f \in X^*$ s.t.

$f(x_0) = \|x_0\| \neq 0 \Rightarrow f \neq 0$

Cor 3 $X \ni x \neq y \Rightarrow \exists f \in X^*$ s.t. $f(x) \neq f(y)$

Pf 对 $x_0 = x - y$ 应用 Cor 2

Cor 3' $\forall f \in X^*$, $f(x) = f(y) \Rightarrow x = y$

特别地 $\forall f \in X^*$, $f(x) = 0 \Rightarrow x = 0$

例: X — Banach 空间

$\{x_k\}_{k=1}^{\infty} \subset X$ s.t. $\sum_{k=1}^{\infty} \|x_k\| < \infty$ (称级数 $\sum_{k=1}^{\infty} x_k$ 绝对收敛)

则 $\forall \sigma: \mathbb{N} \to \mathbb{N}$ 双射

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

Pf. $\forall f \in X^*$

$$\sum_{k=1}^{\infty} |f(x_k)| \le \|f\| \sum_{k=1}^{\infty} \|x_k\| < \infty$$

$\Rightarrow$ $\sum_{k=1}^{\infty} f(x_k)$ 是绝对收敛的数项级数, 重排不变.

$$\Rightarrow \sum_{k=1}^{\infty} f(x_{\sigma(k)}) = \sum_{k=1}^{\infty} f(x_k).$$

$$\Rightarrow f\left(\sum_{k=1}^{\infty} x_{\sigma(k)}\right) = f\left(\sum_{k=1}^{\infty} x_k\right), \quad \forall f \in X^*$$

$$\overset{\text{Cor 3'}}{\Longrightarrow} \sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_k$$

Cor 4 $\|x\| = \sup_{\substack{f \in X^* \\ \|f\| = 1}} |f(x)|$

Pf $\forall f \in X^*$ with $\|f\| = 1$,

$$|f(x)| \le \|f\| \, \|x\| = \|x\|$$

$$\Rightarrow \sup_{\substack{f \in X^* \\ \|f\| = 1}} |f(x)| \le \|x\|$$

另一方面，$\exists\, f \in X^*$，$\|f\| = 1$ s.t. $f(x) = \|x\|$.

<u>Thm</u> $(X, \|\cdot\|)$

$M \hookrightarrow X$

$x_0 \in X$ s.t. $d = \operatorname{dist}(x_0, M) > 0$

$\Rightarrow \exists\, f \in X^*$, $\|f\| = 1$ s.t.

$f(M) = \{0\}$ 而 $f(x_0) = d$

<u>Pf</u> 令 $\tilde{M} \overset{\text{def}}{=} M \oplus \operatorname{span}\{x_0\}$

定义 $f_0 : \tilde{M} \to \mathbb{K}$

$x = y + \lambda x_0 \mapsto \lambda d$

$\Rightarrow f_0(M) = \{0\}$, $f_0(x_0) = d$.

$\forall\, x = y + \lambda x_0$, with $y \in M$, $\lambda \in \mathbb{K}$,

若 $\lambda = 0$. 则 $f_0(x) = 0$

若 $\lambda \neq 0$.

$$
\begin{aligned}
|f_0(x)| &= |\lambda| \operatorname{dist}(x_0, M) \\
&\le |\lambda| \left\| x_0 + \frac{y}{\lambda} \right\| \\
&= \| y + \lambda x_0 \| = \|x\|
\end{aligned}
$$

$\Rightarrow f_0 \in \tilde{M}^*$ 且 $\|f_0\| \le 1$.

$\overset{\text{HBT}}{\Longrightarrow} \exists\, f \in X^*$ s.t.

$$
\begin{cases}
f|_{\tilde{M}} = f_0 & \Rightarrow f(M) = \{0\}, \quad f_0(x_0) = d \\
\|f\| = \|f_0\| & \Rightarrow \|f\| \le 1
\end{cases}
$$

再证 $\|f\| \geq 1$.

$$d = \inf_{y \in M} \|x_0 - y\|$$

$\Rightarrow \forall n, \exists y_n \in M$ s.t.

$$\|x_0 - y_n\| < d + \frac{1}{n}$$

$$\Rightarrow \frac{|f(x_0 - y_n)|}{\|x_0 - y_n\|} = \frac{|f(x_0)|}{\|x_0 - y_n\|} > \frac{d}{d + \frac{1}{n}} \to 1 \quad \text{as } n \to \infty$$

$$\Rightarrow \sup_n \frac{|f(x_0 - y_n)|}{\|x_0 - y_n\|} \geq 1$$

$\Rightarrow \|f\| \geq 1$.

Thm $(X, \|\cdot\|)$

$M \subset X$, $0 \neq x_0 \in X$

$$x_0 \in \overline{\text{span}(M)} \iff f(x_0) = 0, \ \forall f \in X^* \text{ with } f(M) = \{0\}$$

Pf 1° "$\Rightarrow$"

设 $x_0 \in \overline{\text{span}(M)}$

$\forall f \in X^*$ with $f(M) = \{0\}$

$\overset{\text{线性}}{\Longrightarrow} f(\text{span}(M)) = \{0\}$

$\overset{\text{连续}}{\Longrightarrow} f(\overline{\text{span}(M)}) = \{0\}$

$\Rightarrow f(x_0) = 0$

2° "$\Leftarrow$"

假设 $x_0 \notin \overline{\text{span}(M)}$

$\Rightarrow d = \text{dist}(x_0, \overline{\text{span}(M)}) > 0$

$\overset{\text{前一Thm}}{\Longrightarrow}$ $\exists f \in X^*$ with $\|f\| = 1$ s.t.

$f(\overline{\mathrm{span}(M)}) = \{0\}$ 而 $f(x_0) = d$.

矛盾.

Def X — 向量空间

$C \subset X$

(i) 如果 $\forall x, y \in C$, $\forall t \in [0,1]$

$tx + (1-t)y \in C$

则称 C 是凸的

(ii) 如果 $-C = C$, 则称 C 对称

(iii) 如果 $\forall x \in X$, $\exists t > 0$, s.t. $\frac{x}{t} \in C$, 则称

C 是吸收的

Prop 任一族凸集之交仍是凸集

Def $\mathrm{conv}(A) \overset{\text{def}}{=} \bigcap_{\substack{C \supset A \\ C \text{ convex}}} C$ 称为 A 的凸包

(convex hull)

Prop $\mathrm{conv}(A) = \left\{ \underbrace{\sum_{k=1}^{n} \lambda_k x_k}_{\text{凸组合}} \;\middle|\; \begin{array}{l} x_1, \cdots, x_n \in A \\ \lambda_1, \cdots, \lambda_n \in [0,1], \sum_{k=1}^{n} \lambda_k = 1 \\ n \in \mathbb{N} \end{array} \right\}$

Def X — 向量空间

C — 包含 0 的凸集

Define $P_C : X \to [0, +\infty]$ by

$$P_C(x) \overset{\text{def}}{=} \inf\left\{ t > 0 \;\middle|\; \frac{x}{t} \in C \right\}$$

称为 C 的 Minkowski 泛函(或度规, gauge)

Rmk: $P_C(x) = +\infty \iff \left\{ t > 0 : \frac{x}{t} \in C \right\} = \phi$

Prop (i) $P_C(0) = 0$

(ii) (正齐次性) $P_C(tx) = t P_C(x)$, $\forall x \in X$, $\forall t > 0$

(iii) (次可加性) $P_C(x+y) \le P_C(x) + P_C(y)$ $\forall x, y \in X$

(Note: P_C 不一定是 sublinear functional, ∵ 可能取 $+\infty$)

Pf (iii) 不妨设 $P_C(x), P_C(y) \in \mathbb{R}$

$\forall \varepsilon > 0$,

$$\lambda \overset{\text{def}}{=} P_C(x) + \varepsilon/2$$

$$\mu \overset{\text{def}}{=} P_C(y) + \varepsilon/2$$

$$\Rightarrow \frac{x}{\lambda}, \frac{y}{\mu} \in C$$

$$\Rightarrow \frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu} \cdot \frac{x}{\lambda} + \frac{\mu}{\lambda+\mu} \cdot \frac{y}{\mu} \in C$$

$$\Rightarrow \lambda + \mu \ge P_C(x+y)$$

$$\Rightarrow P_C(x+y) \le P_C(x) + P_C(y) + \varepsilon$$

$$\Rightarrow P_C(x+y) \le P_C(x) + P_C(y)$$

Def X — 复向量空间

C — 包含 0 的凸集

如果 $\forall x \in C$, $\forall \theta \in \mathbb{R}$, $e^{i\theta} x \in C$, 则称 C 均衡

Prop 复向量空间中每个均衡、吸收凸集都决定一个半范数.

Pf 吸收 $\Rightarrow$ P_C 半次线性泛函

均衡 $\Rightarrow$ P_C 齐次

HW: Ex. 1.5.1

Ex. 2.4.5–2.4.7.