第十四讲 (2024.10.23)

纲推理 (category argument)

Thm (Banach, 1931)

$\{C[0,1]$ 中处处不可微的函数$\}$ 是第二纲集

Pf $X \overset{\text{def}}{=} C[0,1]$

$A \overset{\text{def}}{=} \{f \in C[0,1]:\ f \text{ 处处不可微}\}$

只需证：$X \setminus A$ 是第一纲集

$X \setminus A = \{f \in C[0,1]:\ f \text{ 至少在一点可微}\}$

$$A_n \overset{\text{def}}{=} \left\{ f \in C[0,1]:\ \exists\, t \in \left[0, 1-\tfrac{1}{n}\right],\ \text{s.t.} \sup_{h \in [-\frac{1}{n}, \frac{1}{n}]} \left| \frac{f(t+h)-f(t)}{t} \right| \le n \right\}$$

$n = 2, 3, \cdots$

1° $A_n \subset A_{n+1}$.

2° 如果 f 在某点可微，则 $f \in A_n$ for some n

$\Rightarrow\quad X \setminus A \subset \bigcup_{n=2}^{\infty} A_n$

故只需证：A_n 是疏集，$\forall n$.

(i) A_n 闭

设 $A_n \ni f_k \to f$ (i.e. $f_k \rightrightarrows f$)

对每个 f_k，$\exists\, t_k \in \left[0, 1-\frac{1}{n}\right]$ s.t.

$$|f_k(t_k - h) - f_k(t_k)| \le n|h|, \quad \forall h \in \left[-\tfrac{1}{n}, \tfrac{1}{n}\right]$$

$\{t_k\}_{n=1}^{\infty}$ 有子列 $t_{k_j} \to t_0 \in \left[0, 1-\frac{1}{n}\right]$.

$$\Rightarrow \quad |f(t_0+h) - f(t_0)|$$

$$\leq |f(t_0+h) - f(t_{k_j}+h)| + |f(t_{k_j}+h) - f_{k_j}(t_{k_j}+h)|$$

$$+ |f_{k_j}(t_{k_j}+h) - f_{k_j}(t_{k_j})| + |f_{k_j}(t_{k_j}) - f(t_{k_j})|$$

$$+ |f(t_{k_j}) - f(t_0)|$$

$$= I_1 + \cdots + I_5$$

f 连续 $\Rightarrow \quad I_1, I_5 < \frac{|h|}{4}\varepsilon$, 当 j 充分大

$f_{k_j} \rightrightarrows f \quad \Rightarrow \quad I_2, I_4 < \frac{|h|}{4}\varepsilon$, 〃

而 $\quad I_3 < n|h| \quad (\because f_{k_j} \in A_n)$

$$\Rightarrow \quad |f(t_0+h) - f(t_0)| \leq (n+\varepsilon)|h|$$

$$\Rightarrow \quad |f(t_0+h) - f(t_0)| \leq n|h|$$

$$\Rightarrow \quad f \in A_n$$

(2) $\quad \mathrm{int}(A_n) = \phi$

只需证: $\forall f \in A_n, \ \forall \varepsilon > 0, \ B(f, \varepsilon) \setminus A_n \neq \phi$

首先, $\exists\, p \in P[0,1]$ s.t.

$$\|f - p\| < \varepsilon/2 \quad \text{(by density)}$$

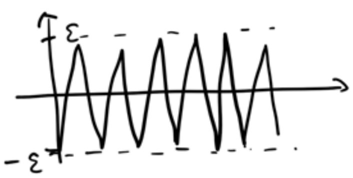
令 $$M \overset{\text{def}}{=} \max_{t\in[0,1]} |p'(t)|$$

$$\Rightarrow \quad |p(t+h) - p(t)| \leq M|h|, \quad \forall t \in [0, 1-\tfrac{1}{n}], \ \forall h \in [-\tfrac{1}{n}, \tfrac{1}{n}]$$

其次, $\exists\, g \in C[0,1]$ s.t.

(i) 分段线性

(ii) $\|g\| < \varepsilon/2$



(iii) 各段斜率绝对值 $> M + n$

$\Rightarrow \quad p + g \in B(f, \varepsilon)$

$\left(\|(p+g) - f\| \leq \|f - p\| + \|g\| < \varepsilon \right)$

但 $\quad p + g \notin A_n$

(除有限个之外， $|(p+g)'(t)| \geq |g'(t)| - |p'(t)| > n$)

Banach 空间三大定理 $\begin{cases} \text{Hahn-Banach} \\ \text{开映射 Thm} \ (\Leftrightarrow \text{闭图像 Thm}) \\ \text{一致有界原理} \end{cases}$

Thm (UBP = Uniform Boundedness Principle)

X — Banach 空间

Y — 赋范空间

$\mathcal{F} \subset \mathcal{L}(X, Y)$

$$\underbrace{\forall x \in X, \ \sup_{T \in \mathcal{F}} \|Tx\| < \infty}_{\text{逐点有界}} \quad \Longrightarrow \quad \underbrace{\sup_{T \in \mathcal{F}} \|T\| < \infty}_{\text{一致有界}}$$

等价地，

$$\sup_{T \in \mathcal{F}} \|T\| = \infty \quad \Longrightarrow \quad \exists\, x_0 \in X \ \text{ s.t. } \sup_{T \in \mathcal{F}} \|Tx_0\| = \infty$$

Pf 令

$$F_n \stackrel{\text{def}}{=} \left\{ x \in X : \sup_{T \in \mathcal{F}} \|Tx\| \leq n \right\}$$

$$= \bigcap_{T \in \mathcal{F}} \left\{ x \in X : \|Tx\| \leq n \right\} \qquad \text{闭}$$

$\forall x \in X, \ \sup\limits_{T \in \mathcal{F}} \|Tx\| < \infty$

$\Rightarrow \ X = \bigcup\limits_{n=1}^{\infty} F_n$

$\overset{\text{BCT}}{\Longrightarrow} \ \exists n_0 \ \text{s.t.} \ F_{n_0}$ 有内点

$\Rightarrow \ \exists B(x_0, r) \subset F_{n_0}$

$\Rightarrow \ \|T(x_0 + rx)\| \leq n_0, \ \forall x \in B(0,1), \ \forall T \in \mathcal{F}$

$\Rightarrow \ \|T(rx)\| \leq n_0 + \|Tx_0\| \leq 2n_0$

$\Rightarrow \ \|Tx\| \leq \dfrac{2n_0}{r}, \ \forall x \in B(0,1), \ \forall T \in \mathcal{F}$

$\Rightarrow \ \sup\limits_{T \in \mathcal{F}} \underbrace{\sup\limits_{x \in B(0,1)} \|Tx\|}_{= \|T\|} \leq \dfrac{2n_0}{r}$

<u>Thm</u> (Banach-Steinhauss)

X — Banach空间

Y — 赋范空间

$T, T_n \in \mathcal{L}(X, Y), \ n = 1, 2, \cdots$

$\overline{M} = X$

$Tx_n \to Tx, \ \forall x \in X \ \Rightarrow \begin{cases} \sup\limits_n \|T_n\| < \infty \\ T_n x \to Tx, \ \forall x \in M \end{cases}$

<u>Pf</u> 1° "$\Rightarrow$"

逐点收敛 $\Rightarrow$ 逐点有界 $\overset{\text{UBP}}{\Longrightarrow}$ 一致有界.

2° "$\Leftarrow$"

令 $C \overset{\text{def}}{=} \sup\limits_n \|T_n\|$

$$\overline{M} = X \Rightarrow \forall x \in X, \forall \varepsilon > 0, \exists y \in M \text{ s.t. } \|x - y\| < \frac{\varepsilon}{4(\|T\| + C)}$$

$$\Rightarrow \|T_n x - Tx\|$$

$$\leq \underbrace{\|T_n x - T_n y\|}_{\substack{\leq C\|x-y\| \\ < \varepsilon/4}} + \underbrace{\|T_n y - Ty\|}_{\substack{< \varepsilon/2 \\ \text{当 } n \text{ 充分大}}} + \underbrace{\|Ty - Tx\|}_{\substack{\leq \|T\|\|x-y\| \\ < \varepsilon/4}}$$

$< \varepsilon$，当 n 充分大.

Thm X, Y — Banach 空间

$T_n \in \mathcal{L}(X, Y)$，$n = 1, 2, \ldots$

如果 $\forall x \in X$，$\lim\limits_{n\to\infty} T_n x$ 存在，定义 $T: X \to Y$, $x \mapsto \lim\limits_{n\to\infty} T_n x$

则 $T \in \mathcal{L}(X, Y)$ 且

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|$$

(HW. Ex. 2.3.7 – 2.3.9)

例：Fourier 级数的点态收敛.

$$S_n f(x) \stackrel{\text{def}}{=} \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k x}$$

Thm (Du Bois-Reymond, 1876)

$\exists f \in C(\mathbb{T})$ s.t. $\{S_n f(0)\}_{n=1}^{\infty}$ 发散

Pf $\quad S_n f(x) = (f * D_n)(x) = \int_{-1/2}^{1/2} f(t) D_n(x-t)\, dt$

with $\quad D_n(t) = \sum_{k=-n}^{n} e^{2\pi i k t} = \dfrac{\sin[(2n+1)\pi t]}{\sin(\pi t)}$

定义 $\quad T_n : C(\mathbb{T}) \to \mathbb{R}$

$\qquad f \mapsto S_n f(0)$ （注意 D_n 是实值函数. 如果 f 实值, 则 $S_n f$ 也是）

$$\Rightarrow \quad |T_n f| = \left| \int_{-1/2}^{1/2} f(t) D_n(-t)\, dt \right| \leq \|D_n\|_1 \cdot \|f\|$$

$$\Rightarrow \quad T_n \in C(\mathbb{T})^* \quad \underline{\text{且}} \quad \|T_n\| \leq \|D_n\|_1$$

<u>Claim</u> $\quad \|T_n\| = \|D_n\|_1$

D_n 在 $[-\frac{1}{2}, \frac{1}{2})$ 内只有有限个零点

$\Rightarrow$ $\operatorname{sgn} D_n$ 只有有限个间断点

$\forall \varepsilon > 0$, $\exists f_\varepsilon \in C(\mathbb{T})$ s.t.

(i) 分段线性

(ii) $\|f_\varepsilon\| = 1$

(iii) $f_\varepsilon = \operatorname{sgn} D_n$ on $[-\frac{1}{2}, \frac{1}{2}) \setminus I_\varepsilon$ with $|I_\varepsilon| < \frac{\varepsilon}{4n+3}$

$$\Rightarrow \quad |T_n f_\varepsilon| = \left| \int_{-1/2}^{1/2} f_\varepsilon(t) D_n(t)\, dt \right|$$

$$\geq \int_{[-\frac{1}{2}, \frac{1}{2}) \setminus I_\varepsilon} |D_n(t)|\, dt - \int_{I_\varepsilon} |D_n(t)|\, dt$$

$$\geq \|D_n\|_1 - 2 \int_{I_\varepsilon} |D_n(t)|\, dt$$

$$> \|D_n\|_1 - \varepsilon \qquad (\|D_n\|_\infty \leq 2n+1)$$

$$\Rightarrow \quad \|T_n\| \geq \frac{|T_n f_\varepsilon|}{\|f_\varepsilon\|} > \|D_n\|_1 - \varepsilon$$

$$\Rightarrow \quad \|T_n\| \geq \|D_n\|_1$$

而 $\quad \|D_n\|_1 = 2 \int_0^{1/2} \left| \frac{\sin[(2n+1)\pi t]}{\sin(\pi t)} \right| dt$

$$\geq 2 \int_0^{1/2} \left| \frac{\sin[(2n+1)\pi t]}{\pi t} \right| dt$$

$$\overset{x=(2n+1)\pi t}{=} \frac{2}{\pi} \int_0^{\frac{\pi}{2}(2n+1)} \left| \frac{\sin x}{x} \right| dx$$

$$\longrightarrow +\infty \quad \text{as} \quad n \to \infty$$

$$\Longrightarrow \quad \sup_n \|T_n\| = \infty$$

$$\overset{\text{UBP}}{\Longrightarrow} \quad \exists f \in C(\mathbb{T}) \quad \text{s.t.}$$

$$\sup_n |T_n f| = \infty$$

$$\Longrightarrow \quad \limsup_{n\to\infty} |S_n f(0)| = \infty$$

$$\Longrightarrow \quad \{S_n f(0)\}_{n=1}^{\infty} \text{ 发散}$$