第十讲 (2024.10.9)

Def $(X, \langle\cdot,\cdot\rangle)$

$\{e_\alpha\}_{\alpha\in I}$ — O.N.S.

如果 $\forall x \in X$,

$$x = \sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha ,$$

则称 $\{e_\alpha\}_{\alpha\in I}$ 为 X 的一个 O.N.B.

上次课已证明: $\forall x \in X$, $\tilde{I} \overset{def}{=} \{\alpha \in I : \langle x, e_\alpha\rangle \neq 0\}$ 至多可数

令 $\tilde{I} = \{\alpha_k\}_{k=1}^\infty$

$$\sum_{\alpha\in I} \langle x, e_\alpha\rangle e_\alpha \overset{def}{=} \sum_{k=1}^\infty \langle x, e_{\alpha_k}\rangle e_{\alpha_k}$$

Q: well-defined ?

Lem H — Hilbert 空间

$\{e_k\}_{k=1}^\infty$ — O.N.S.

$$M \overset{def}{=} \overline{\text{span}\,\{e_k\}_{k=1}^\infty} \implies \forall x \in H,\ \sum_{k=1}^\infty \langle x, e_k\rangle e_k \in M$$

$$\text{且} \quad \sum_{k=1}^\infty \langle x, e_k\rangle e_k = P_M x$$

Pf $\displaystyle\sum_{k=1}^\infty |\langle x, e_k\rangle|^2 \leq \|x\|^2$ (Bessel)

$$\implies \left\| \sum_{k=n}^m \langle x, e_k\rangle e_k \right\|^2 = \sum_{k=n}^m |\langle x, e_k\rangle|^2$$

$$\to 0 \quad \text{as } n, m \to \infty$$

$$\Rightarrow \quad \left\{ \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\}_{n=1}^{\infty} \text{ 为 } H \text{ 中 Cauchy 列.}$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_k \rangle e_k \in H$$

进而,

$$\left\langle x - \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, \; e_m \right\rangle = \langle x, e_m \rangle - \langle x, e_m \rangle = 0, \quad \forall m \in \mathbb{N}$$

$$\Rightarrow \quad x - \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k \in M^{\perp}$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k = P_M x \quad (\text{by defn. 最佳逼近})$$

Cor 对 $\mathbb{N}$ 上任一置换 σ

$$\sum_{k=1}^{\infty} \langle x, e_{\sigma(k)} \rangle e_{\sigma(k)} = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

Pf 令

$$M \overset{\text{def}}{=} \overline{\text{span}\,\{e_k\}_{k=1}^{\infty}}$$

$$\tilde{M} \overset{\text{def}}{=} \overline{\text{span}\,\{e_{\sigma(k)}\}_{k=1}^{\infty}}$$

$$\Rightarrow \quad M = \tilde{M}$$

$$\Rightarrow \quad \text{LHS} = P_{\tilde{M}} x = P_M x = \text{RHS}$$

Thm H — Hilbert 空间

$\{e_\alpha\}_{\alpha \in I}$ — O.N.S.

$\forall x \in H$, $\sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha \in H$

则

$$\left\| x - \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha \right\|^2 = \|x\|^2 - \sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2$$

Pf 设 $\{\alpha \in I : \langle x, e_\alpha \rangle \neq 0\} = \{\alpha_k\}_{k=1}^{\infty}$

$$\Rightarrow \sum_{\alpha \in I} \langle x, e_\alpha \rangle e_\alpha = \sum_{k=1}^{\infty} \langle x, e_{\alpha_k} \rangle e_{\alpha_k}$$

而

$$\left\| x - \sum_{k=1}^{n} \langle x, e_{\alpha_k} \rangle e_{\alpha_k} \right\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_{\alpha_k} \rangle|^2$$

$$\overset{n\to\infty}{\Longrightarrow} \left\| x - \sum_{k=1}^{\infty} \langle x, e_{\alpha_k} \rangle e_{\alpha_k} \right\|^2 = \|x\|^2 - \sum_{k=1}^{\infty} |\langle x, e_{\alpha_k} \rangle|^2$$

Thm H — Hilbert 空间

$S = \{e_\alpha\}_{\alpha \in I}$ — O.N.S.

则 S 是 O.N.B. $\Leftrightarrow$ $S^\perp = \{0\}$ （完备）

$\Leftrightarrow \forall x \in H, \ \|x\|^2 = \sum_{\alpha \in I} |\langle x, e_\alpha \rangle|^2$

(Parseval Identity)

Pf 1° O.N.B. $\Rightarrow$ P.I.

由 引 - Cor

2° P.I. $\Rightarrow$ 完备

假设 $S^\perp \neq \{0\}$

$\Rightarrow \exists\ 0 \neq x_0 \in H$ s.t.

$\langle x_0, e_\alpha \rangle = 0, \quad \forall \alpha \in I.$

$$\Rightarrow \sum_{\alpha \in I} |\langle x_0, e_\alpha \rangle|^2 = 0$$

$$\overset{P.I.}{\Longrightarrow} \|x_0\|^2 = 0, \quad 矛盾$$

3° 完备 $\Rightarrow$ O.N.B.

假设 $S^{\perp} = \{0\}$ 但 $\underbrace{S \text{ 不是 O.N.B.}}_{\exists x_0 \in H \text{ s.t. } \sum_{\alpha \in I} \langle x_0, e_\alpha \rangle e_\alpha \neq x_0}$

而 $\forall \beta \in I$,

$$\left\langle x_0 - \sum_{\alpha \in I} \langle x_0, e_\alpha \rangle e_\alpha ,\ e_\beta \right\rangle = \langle x_0, e_\beta \rangle - \langle x_0, e_\beta \rangle$$
$$= 0$$

$$\Rightarrow \quad x_0 - \sum_{\alpha \in I} \langle x_0, e_\alpha \rangle e_\alpha \perp S$$

与 $S^{\perp} = \{0\}$ 矛盾.

例: ℓ^2 中

$(\{e_n\}_{n=1}^{\infty})^{\perp} = \{0\}$ （如果 $x = (x_1, x_2, \dots) \perp e_n$ $\Rightarrow x_n = 0$

$\Rightarrow \{e_n\}_{n=1}^{\infty}$ 是 O.N.B.

Rmk: $\{e_n\}_{n=1}^{\infty}$ 不是 Hamel 基

Cor 非平凡 Hilbert 空间一定有 O.N.B.

Thm (Gram-Schmidt 正交化)

$(X, \langle \cdot, \cdot \rangle)$

$\{x_n\}_{n=1}^{\infty}$ 线性无关 $\Rightarrow \exists \{e_n\}_{n=1}^{\infty}$ O.N.S. s.t.

$\forall n$,

$$\operatorname{span}\{e_k\}_{k=1}^{n} = \operatorname{span}\{x_k\}_{k=1}^{n}$$

Pf 令 $y_1 = x_1$ $\qquad e_1 = \dfrac{y_1}{\|y_1\|}$

$$y_2 = x_2 - \langle x_2, e_1 \rangle e_1 \qquad e_2 = \frac{y_2}{\|y_2\|}$$

$$\vdots$$

$$y_n = x_n - \sum_{k=1}^{n-1} \langle x_n, e_k \rangle e_k, \qquad e_n = \frac{y_n}{\|y_n\|}$$

$$\vdots$$

Thm H — Hilbert 空间

H 可分 $\Longleftrightarrow$ H 有至多可数的 O.N.B.

Pf $1°$ "$\Rightarrow$"

Case 1 $\dim H < \infty$

取基, Hamel $\overset{G-S}{\Longrightarrow}$ O.N.B.

Case 2 $\dim H = \infty$

可分 $\Rightarrow$ $\exists\, A = \{x_n\}_{n=1}^{\infty} \subset H$ s.t. $\overline{A} = H$.

Claim $\exists\, B = \{y_k\}_{k=1}^{\infty} \subset A$ 线性无关 s.t.

$$\operatorname{span}\{y_k\}_{k=1}^{\infty} = \operatorname{span}\{x_n\}_{n=1}^{\infty}$$

取 $x_{n_1} = x_1$

$x_{n_2} \in A$ s.t. $x_{n_2} \notin \operatorname{span}\{x_{n_1}\}$

$$\vdots$$

$x_{n_k} \in A$ s.t. $x_{n_k} \notin \operatorname{span}\{x_{n_1}, \cdots, x_{n_{k-1}}\}$

$$\vdots$$

令 $y_k = x_{n_k}$, $k = 1, 2, \cdots$, $B \overset{\text{def}}{=} \{y_k\}_{k=1}^{\infty}$

$$\Rightarrow \quad \forall x_k \in A. \quad x_k \in \operatorname{span}\{y_1, \cdots, y_{k-1}\}$$

(否则 按 y_k 的定义. $k \geq n_k$)

$$\Rightarrow \quad A \subset \operatorname{span} B$$

而 $\overline{\operatorname{span} B} = \overline{\operatorname{span} A} = H$

$\Rightarrow \quad \# B = \infty$ (否则 $\operatorname{span} B$ 是有限维子空间. 是闭子空间 与 $\overline{\operatorname{span} B} = H$ 矛盾)

G-S

$$\Rightarrow \quad \exists \{e_n\}_{n=1}^{\infty} \quad \text{O.N.S.} \quad \text{s.t.}$$

$$\operatorname{span}\{e_n\}_{n=1}^{\infty} = \operatorname{span}\{x_n\}_{n=1}^{\infty}$$

$$\Rightarrow \quad \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}} = \overline{\operatorname{span} A} = H.$$

$$\Rightarrow \quad \left(\{e_n\}_{n=1}^{\infty}\right)^{\perp} = \{0\}$$

$$\Rightarrow \quad \{e_n\}_{n=1}^{\infty} \text{ 是 O.N.B.}$$

2° "$\Leftarrow$"

设 $\{e_n\}_{n=1}^{\infty}$ 是 O.N.B.

令 $\quad M \overset{\text{def}}{=} \operatorname{span}^{\mathbb{Q}} \{x_n\}_{n=1}^{\infty}$

$$\overset{\text{def}}{=} \left\{ \sum_{k=1}^{n} \lambda_k e_k : \lambda_1, \cdots, \lambda_n \in \mathbb{Q} + i\mathbb{Q}. \ n \in \mathbb{N} \right\}$$

$\Rightarrow \quad M$ 可数

(与 $\displaystyle\bigcup_{\substack{I \subset \{e_n\}_{n=1}^{\infty} \\ \# I < \infty}} (\mathbb{Q} + i\mathbb{Q})^{\# I}$ 一一对应)

Claim $\overline{M} = H$

$\forall x \in H, \quad x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$

$\forall \varepsilon > 0, \ \forall n, \ \exists \alpha_n \in \mathbb{Q} + i\mathbb{Q}$ s.t.

$$\left| \alpha_n - \langle x, e_n \rangle \right| < \frac{\varepsilon}{2^{n+1}}$$

$$\Rightarrow \quad \left\| \sum_{n=1}^{N} \langle x, e_n \rangle e_n - \sum_{n=1}^{N} \alpha_n e_n \right\|^2$$

$$= \sum_{n=1}^{N} \left| \langle x, e_n \rangle - \alpha_n \right|^2 < \frac{\varepsilon^2}{4}, \quad \forall N$$

而当 N 充分大时

$$\left\| \sum_{n=1}^{N} \langle x, e_n \rangle e_n - x \right\| < \varepsilon/2$$

$$\Rightarrow \quad \Big\| \underbrace{\sum_{n=1}^{N} \alpha_n e_n}_{\in M} - x \Big\| < \varepsilon$$

HW: Ex. 1.6.11, 1.6.12, 1.6.16