第二讲 (2024.9.4)

Def $X \neq \phi$

如果 $d: X \times X \to \mathbb{R}$ s.t.

(M1) $d(x,y) \geq 0$, $\forall x, y \in X$

$d(x,y) = 0 \iff x = y$

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,y) \leq d(x,z) + d(z,y)$

则称 d 是 X 上一个度量. (X, d) 称度量空间.

这是一个拓扑结构

Def $A \subset X$

$$\mathrm{diam}(A) \overset{def}{=} \sup_{x,y \in A} d(x,y)$$ 称为 A 的直径.

如果 $\mathrm{diam}(A) < \infty$, 则称 A 有界

Def (X, d)

称 $\{x_n\}_{n=1}^{\infty} \subset X$ 收敛是指: $\exists\, x_0 \in X$ s.t.

$d(x_n, x_0) \to 0$ as $n \to \infty$,

记为 $x_n \to x_0$.

HW: 1° 收敛列极限唯一

2° 收敛列有界

例: $C[0,1] \overset{def}{=} [0,1]$ 上连续函数全体.

$$d(f,g) \overset{def}{=} \max_{t \in [0,1]} |f(t) - g(t)|$$

$d(f_n, f) \to 0 \iff f_n \rightrightarrows f$

$\rho_1(f,g) \overset{\text{def}}{=} \int_0^1 |f(t)-g(t)|\,dt$ （L^1 距离）

$$f_n(t) \overset{\text{def}}{=} \begin{cases} -n^3\left(t-\frac{1}{n^2}\right), & t\in\left[0,\frac{1}{n^2}\right] \\ 0, & t\in\left(\frac{1}{n^2},1\right] \end{cases}$$

$\rho_1(f_n,0) = \frac{1}{2}\cdot n\cdot\frac{1}{n^2} = \frac{1}{2n} \to 0$ as $n\to\infty$

但 $d(f_n,0) = n \not\to 0$.

Def (X,d), $x_0\in X$. $r>0$.

开球 $B(x_0,r) \overset{\text{def}}{=} \{x\in X : d(x,x_0) < r\}$

闭球 $\overline{B}(x_0,r) \overset{\text{def}}{=} \{x\in X : d(x,x_0) \leq r\}$

球面 $S(x_0,r) \overset{\text{def}}{=} \{x\in X : d(x,x_0) = r\}$

设 $A\subset X$.

如果 $\forall x\in A$, $\exists r_x>0$ s.t. $B(x,r_x)\subset A$,

则称 A 为开集.

包含 x 的开集称为 x 的邻域.

闭集 $\overset{\text{def}}{=}$ 开集的余集

Prop $\tau \overset{\text{def}}{=} (X,d)$ 中开集全体.

(i) $\phi, X \in \tau$.

(ii) τ 对任意并封闭

(iii) τ 对有限交封闭

Def (X,d)

$A\subset X$, $x_0\in X$

(i) 如果 $\forall \varepsilon>0$, $B(x_0,\varepsilon)\cap A \neq \phi$, 则称 x_0 为 A 的接触点.

(ii) 如果 $\forall \varepsilon > 0$, $B(x_0, \varepsilon) \cap (A \setminus \{x_0\}) \neq \phi$, 则称 x_0 是 A 的聚点或极限点

$(\Leftrightarrow \exists \{x_n\}_{n=1}^{\infty} \subset A \setminus \{x_0\}$ s.t. $x_n \to x_0)$

$\overline{A} \overset{def}{=} \{A$ 的接触点$\}$, 称为 A 的闭包

Rmk $\forall x \in \overline{A}$, $\exists x_n \in A$, $n = 1, 2 \cdots$ s.t. $x_n \to x$.

HW: A 是闭集 $\Leftrightarrow \overline{A} = A$

$\Leftrightarrow \forall \{x_n\}_{n=1}^{\infty} \subset A$

$x_n \to x_0$ implies $x_0 \in A$

Def 如果 $\overline{A} = X$, 称 A 在 X 中稠密, 记 $A \overset{dense}{\subset} X$

Rmk: $A \overset{dense}{\subset} X \Leftrightarrow \forall x \in X$, $\exists \{x_n\}_{n=1}^{\infty} \subset A$

s.t. $x_n \to x$

Def 如果 X 有可数稠密子集, 则称 X 可分

例: $\underbrace{P[0,1]}_{[0,1]\text{上多项式全体}} \overset{dense}{\subset} C[0,1]$

HW: $C[0,1]$ 可分

Def (X, d), (Y, ρ)

称映射 $T: X \to Y$ 在 $x_0 \in X$ 连续是指:

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$d(x, x_0) < \delta \Rightarrow \rho(Tx, Tx_0) < \varepsilon$$

$(\Leftrightarrow Tx \in B_Y(Tx_0, \varepsilon), \forall x \in B(x_0, \delta))$

如果 T 在 X 中每点都连续, 则称 $T: X \to Y$ 连续

Thm 映射 $T: X \to Y$ 连续 $\iff$ $\forall U \overset{\text{open}}{\subset} Y,\ T^{-1}U \overset{\text{open}}{\subset} X$

Pf HW

Thm (Heine)

T 在 x_0 连续 $\iff$ $\forall \{x_n\}_{n=1}^{\infty} \subset X$,

$x_n \longrightarrow x_0$ implies $Tx_n \to Tx_0$

Pf HW

Def: (X, d)

$\{x_n\}_{n=1}^{\infty} \subset X$

如果 $\forall \varepsilon > 0$, $\exists N$ s.t.

$$d(x_m, x_n) < \varepsilon, \quad \forall m, n \geq N$$

则称 $\{x_n\}_{n=1}^{\infty}$ 是 (X, d) 中基本列或 Cauchy 列.

如果 (X, d) 中任一基本列都收敛, 则称 (X, d) 完备.

Rmk: 上述定义可写为

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0$$

例: $(\mathbb{R}, d)$ 完备

$(\mathbb{Q}, d)$ 不完备

$$x_n \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k^2}$$

$$|x_n - x_m| = \sum_{k=n+1}^{m} \frac{1}{k^2} \to 0 \quad \text{as } m, n \to \infty$$

但 $x_n \to \frac{\pi^2}{6} \notin \mathbb{Q}$

HW: 离散度量空间完备.

例: $\Omega = \mathbb{R}^n$ or 可测集 , $1 \le p < \infty$

$$L^p(\Omega) \overset{def}{=} \left\{ f \text{ 可测} : \int_\Omega |f(x)|^p \, dx < \infty \right\}$$

$$d(f, g) \overset{def}{=} \left(\int_\Omega | f(x) - g(x) |^p \, dx \right)^{1/p}$$

$L^p(\Omega)$ 完备 (Riesz - Fischer Thm)

Thm $(C[0,1], d)$ 完备

Pf 设 $\{f_n\}_{n=1}^\infty$ 是 $C[0,1]$ 中 Cauchy 列.

$\Rightarrow \forall \varepsilon > 0, \ \exists N$ s.t.

(*) $$\max_{t \in [0,1]} | f_m(t) - f_n(t) | < \varepsilon , \quad \forall m, n \ge N$$

$\Rightarrow \forall t \in [0, 1]$, $\{f_n(t)\}_{n=1}^\infty$ 是 $\mathbb{R}$ 中 Cauchy 列

$\Rightarrow f(t) \overset{def}{=} \lim_{n \to \infty} f_n(t)$ 存在

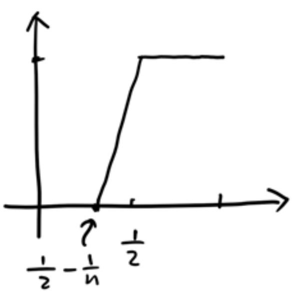
在 (*) 中令 $m \to \infty$

$$\Rightarrow \max_{t \in [0,1]} | f(t) - f_n(t) | \le \varepsilon, \quad \forall n \ge N$$

$\Rightarrow f_n \rightrightarrows f$

$\Rightarrow f \in C[0,1]$ 且 $d(f_n, f) \to 0$

例: $(C[0,1], \rho_1)$ 不完备



令 $$f_n(t) \overset{def}{=} \begin{cases} 0, & t \in [0, \frac{1}{2} - \frac{1}{n}) \\ nt - \frac{n}{2} + 1, & t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ 1, & t \in (\frac{1}{2}, 1] \end{cases}$$

$$\rho_1(f_m, f_n) = \int_0^1 | f_m(t) - f_n(t) | \, dt = \frac{1}{2} \left| \frac{1}{m} - \frac{1}{n} \right|$$

$\to 0$ as $m, n \to \infty$

$\Rightarrow \{f_n\}_{n=1}^{\infty}$ 为 $(C[0,1], \rho_1)$ 中 Cauchy 列)

Claim $\{f_n\}_{n=1}^{\infty}$ 不收敛

假设 $\exists f \in C[0,1]$ s.t. $\rho_1(f_n, f) \to 0$

$$\rho_1(f_n, f) = \int_0^{\frac{1}{2}-\frac{1}{n}} |f(t)|\,dt + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} |f_n(t) - f(t)|\,dt + \int_{\frac{1}{2}}^{1} |1 - f(t)|\,dt$$

$$\to \int_0^{\frac{1}{2}} |f(t)|\,dt + \int_{\frac{1}{2}}^{1} |1 - f(t)|\,dt$$

$$\Rightarrow \quad f(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}) \\ 1, & t \in [\frac{1}{2}, 1] \end{cases}$$

与 f 连续矛盾

APPENDIX TO CHAPTER 5:
When can $\mathbb{R}^n$ be replaced by "metric space"?
by R. Gulliver

In this book we have concentrated much of our attention on concrete metric spaces, especially $\mathbb{R}^n$. The question naturally arises, how general are the results we have obtained? In many exercises we have already asked the reader to verify that some results hold in general metric spaces (see for example p. 100). In the table below are gathered together some of the important results, (including some not formally stated as theorems in the text) and the general contexts in which they are valid are stated. The proofs are, in almost every case, the same as those given in the text. The reader should pick out some of these theorems and verify that this generalization is indeed valid.

Theorem	Valid in Metric spaces?
Chapter 2	
Theorem 1: For all $\varepsilon > 0$ and $x \in \mathbb{R}^n$, $D(x,\varepsilon)$ is open.	Yes.
Theorem 2: (i) the intersection of a finite number of open sets is open; (ii) the union of any collection of open sets is open.	Yes.
Theorem 3: (reverse of Theorem 2 for closed sets).	Yes.
Theorem 4: $A \subset \mathbb{R}^n$ is closed iff all accumulation points of A in $\mathbb{R}^n$ belong to A.	Yes.
Theorem 5: $\mathrm{cl}(A)$ consists of A plus all its accumulation points in $\mathbb{R}^n$.	Yes.
Theorem 6: $x \in \mathrm{bd}(A)$ iff every neighborhood of x in $\mathbb{R}^n$ contains points of A and points of $\mathbb{R}^n \backslash A$.	Yes.
Theorem 7: $x_k \to x$ iff for all $\varepsilon > 0$ there exists N such that if $k > N$ then $\|x_k - x\| < \varepsilon$.	Yes.
Theorem 8: $x_k, x \in \mathbb{R}^n$: $x_k \to x$ iff each sequence of components of x_k converges to the corresponding component of x.	Meaningless in a general metric space.
Theorem 9: $A \subset \mathbb{R}^n$ is closed iff for all sequences $\{x_k\}$, $x_k \in A$ which converge in $\mathbb{R}^n$, the limit is in A.	Yes.
Theorem 10: A sequence $\{x_k\}$ in $\mathbb{R}^n$ converges iff it is a Cauchy sequence.	$\Rightarrow$ yes. $\Leftarrow$ is the definition of a complete metric space;
Theorem 11: For $x_k \in \mathbb{R}^n$: $\sum x_k$ converges iff for all $\varepsilon > 0$ there exists N such that if $k \geqslant N$ and $p \geqslant 0$ then $\|x_k + x_{k+1} + \cdots + x_{k+p}\| < \varepsilon$.	Valid in complete normed space (= Banach space).

Theorem	Valid in Metric spaces?
Theorem 12: $x_k \in \mathbb{R}^n$: If $\sum \|x_k\|$ converges in $\mathbb{R}$ then $\sum x_k$ converges in $\mathbb{R}^n$.	Valid in Banach space.
Theorem 13: (iv) If $\lim\limits_{k\to\infty} \|x_{k+1}\|/\|x_k\|$ exists and is <1 then $\sum x_k$ converges. (Also (v) is valid).	Valid in Banach space.
Baire Category Theorem: The intersection of a countable number of dense open subsets of $\mathbb{R}^n$ is dense in $\mathbb{R}^n$.	Valid in complete metric space.
Theorem: $\mathbb{R}^n$ has a countable dense subset.	This defines a "separable" metric space; not always true. However, $\mathscr{C}(A,\mathbb{R}^m)$ is separable, for $A \subset \mathbb{R}^n$ compact (prove this using the Stone-Weierstrass theorem).
Chapter 3	
Theorem 1: The following are equivalent for $A \subset \mathbb{R}^n$: (i) A is closed and bounded. (ii) A has the Heine-Borel property. (iii) A has the Bolzano-Weierstrass property.	No! However, (ii) and (iii) are equivalent, and each implies (i). If A has (ii), we call it *compact*.
Theorem 2: $\{F_k\}$ a sequence of non-empty compact subsets of $\mathbb{R}^n$ with $F_{k+1} \subset F_k$. Then $\bigcap_{k=1}^{\infty} F_k$ is non-empty.	Yes (using the above definition of compact).
Theorem 3: If A is path-connected then it is connected.	Yes.
Theorem: If A is open $\subset \mathbb{R}^n$ and A is connected, then it is path-connected.	In a normed linear space.
Proposition: $\tilde{A}$ a closed subset of A, A compact $\Rightarrow \tilde{A}$ is compact.	Yes.
Proposition: A a closed subset of $\mathbb{R}^n$, $x \notin A \Rightarrow$ there exists $y \in A$ with $d(x,y) = \inf\{d(x,z) \mid z \in A\}$	No!
Chapter 4	
Theorem 1: For $f: A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, these are equivalent: (i) f is continuous on A. (ii) For each sequence $x_k \to x$, $x_k \in A$, $x \in A$, there holds $f(x_k) \to f(x)$. (iii) For all open sets $U \subset \mathbb{R}^m, f^{-1}(U)$ is a relatively open subset of A. (iv) For all closed sets $K \subset \mathbb{R}^m, f^{-1}(K)$ is a relatively closed subset of A.	Yes (replace A by one metric space, $\mathbb{R}^m$ by another metric space)

(*continued*)

Theorem	Valid in Metric spaces?
Theorem 2: $A \subset \mathbb{R}^n$ and $f: A \to \mathbb{R}^m$ continuous. Then (i) If $K \subset A$ is connected, then $f(K)$ is connected. (ii) If $K \subset A$ is compact, then $f(K)$ is compact.	Yes.
Theorem 3: $A \subset \mathbb{R}^n$, $f: A \to \mathbb{R}^m$; $B \subset f(A) \subset \mathbb{R}^m$, $g: B \to \mathbb{R}^p$. If f and g are continuous then $g \circ f: A \to \mathbb{R}^p$ is also continuous.	Yes.
Theorem 4: Sums and scalar products of continuous functions are again continuous.	In a normed space.
Theorem 5: $A \subset \mathbb{R}^n$ compact, $f: A \to \mathbb{R}$ continuous. Then $f(A)$ is bounded and contains its sup and inf.	Yes.
Theorem 6: $A \subset \mathbb{R}^n$ connected, $f: A \to \mathbb{R}$ continuous. For any $x, y \in A$ and $c \in \mathbb{R}$ with $f(x) < c < f(y)$, there exists $z \in A$ such that $f(z) = c$.	Yes.
Theorem 7 (Heine's Theorem): $A \subset \mathbb{R}^n$ compact, $f: A \to \mathbb{R}^m$ continuous. Then f is uniformly continuous on A.	Yes.
Chapter 5	
Theorem 1: $f_k \to f$ uniformly, $f_k, f: A \to \mathbb{R}^m$: $A \subset \mathbb{R}^n$. If each f_k is continuous then f is continuous.	Yes..
Theorem 3 (Weierstrass M-test): $A \subset \mathbb{R}^n g_k: A \to \mathbb{R}^m$, $\|g_k\|_{\sup} \leqslant M_k$ and $\sum M_k$ converges. Then $\sum g_k$ converges uniformly.	A may be any metric space; $\mathbb{R}^m$ must be replaced by a Banach space.
Theorem 8: For $A \subset \mathbb{R}^n$, $\mathscr{C}_b(A,\mathbb{R}^m)$ is a Banach space.	A any metric space; $\mathbb{R}^m$ must be a Banach space.
Theorem 9 (Arzela-Ascoli): $A \subset \mathbb{R}^n$ compact, $B \subset \mathscr{C}(A,\mathbb{R}^m)$. B is compact iff B is closed, bounded, and equicontinuous.	A may be any compact metric space, but $\mathbb{R}^m$ must be $\mathbb{R}^m$.
Theorem 12 (Stone-Weierstrass): $A \subset \mathbb{R}^n$ compact, $B \subset \mathscr{C}(A,\mathbb{R})$. If B is an algebra which separates points and if the constant functions are included in B, then B is dense.	A may be any compact metric space.

Further results on metric spaces:

Theorem: If X is a complete metric space, A a closed subset of X, then A is a complete metric space.

Definition: A metric space X is *totally bounded* if for all $\varepsilon > 0$ there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $X \subset \bigcup_{i=1}^{n} D(x_i,\varepsilon)$.

Theorem: Let X be a metric space. X is compact iff X is complete and totally bounded.