作业题:

2.4.1. (1) $p(2\theta) = p(\theta) = 2p(\theta) \Rightarrow p(\theta) = 0$.

(2) $p(\theta) = p(x + (-x)) \le p(x) + p(-x) \Rightarrow p(-x) \ge -p(x)$.

(3) 若 $x_0 = 0$. 则可任取 $x_1 \neq 0$.

∴只需考虑 $x_0 \neq 0$. 令 $X_0 = \text{span}\{x_0\}$.

令 $f(\lambda x_0) = \lambda p(x_0)$.

∴对 $\lambda \ge 0$. $f(\lambda x_0) = \lambda p(x_0) = p(\lambda x_0)$.

对 $\lambda < 0$ $f(\lambda x_0) = -(-\lambda) p(x_0) = -p(-\lambda x_0) \le p(\lambda x_0)$.

∴由HBT. 延拓存在 □

2.4.2. 正齐次性: $p(\lambda x) = \limsup\limits_{n\to+\infty} \lambda a_n = \lambda \limsup\limits_{n\to+\infty} a_n = \lambda p(x)$ for $\forall \lambda > 0$.

次可加性: $p(x+y) = \limsup\limits_{n\to+\infty}(a_n + b_n)$ (本题应将 X 改为有界数列全体).

令 $a = \limsup\limits_{n\to+\infty} a_n$. $b = \limsup\limits_{n\to+\infty} b_n$.

对 $\forall c$ 为 $a_n + b_n$ 的极限点. 即 ∃子列 $a_{n_k} + b_{n_k} \to c$.

取 a_{n_k} 的子列 $a_{n_{k_l}} \to \tilde{a}$ ∴ $\tilde{a} \le a$ 由 $a_{n_{k_l}} + b_{n_{k_l}} \to c$

∴ $b_{n_{k_l}} \to c - \tilde{a} \Rightarrow c - \tilde{a} \le b$ ∴ $c = \tilde{a} + c - \tilde{a} \le a + b$.

∴对 c 取 sup $\Rightarrow \limsup\limits_{n\to+\infty}(a_n + b_n) \le a + b$ □

2.4.3. 由 $p(x_0)\neq 0$. p 为半范数 $\Rightarrow x_0\neq 0$. 令 $X_0=\operatorname{span}\{x_0\}$.

$\therefore$ 定义 $f(\lambda x_0)=\lambda$.

$\therefore |f(\lambda x_0)| = |\lambda| = \dfrac{p(\lambda x_0)}{p(x_0)}$.

由于 $\dfrac{p(x)}{p(x_0)}$ 同样为半范数 由 HBT. 延拓存在 $\square$.

1.5.1. (1): $\Rightarrow$ $x\in \mathring{E}$ 若 $x=\theta \Rightarrow p(x)=0$.

若 $x\neq 0$. $\exists r>0$ s.t. $B_r(x)\subseteq E$. 即 $x+\frac{r}{2}\frac{x}{|x|}\in E$.

(且不妨设 $r<|x|$).

$\therefore p(x)\leq \dfrac{1}{1+\frac{r}{2}\frac{1}{|x|}}<1$.

$\Leftarrow$. 若 $p(x)<1$. 由连续性 $\exists \varepsilon,\delta>0$ s.t. $p(y)<1-\varepsilon$ 对 $\forall y\in B(x,\delta)$

$\therefore$ 对 $\forall y\in B(x,\delta)$. 取 $0<a_y<1-\frac{1}{2}\varepsilon$.

s.t. $\dfrac{y}{a_y}\in E$ $\because 0\in E \Rightarrow y\in E$.

$\therefore B(x,\delta)\subseteq E \Rightarrow x\in E^{\circ}$

(2): $\overline{E^{\circ}}\subseteq \overline{E}$. 显然.

而 $\overline{E^{\circ}} = \overline{\{x\mid p(x)<1\}} = \{x\mid p(x)\leq 1\}$. (由 p 的定义).

对 $\forall x\in \overline{E}$ $\exists x_n\to x$. $x_n\in E$ 由 $x_n\in E \Rightarrow p(x_n)\leq 1$

$\therefore$ 由连续 $\Rightarrow p(x)\leq 1 \Rightarrow x\in \overline{E^{\circ}} \Rightarrow \overline{E}\subseteq \overline{E^{\circ}}$ $\square$.

2.4.5. 若 $x\in X_0$ 则 $\rho(x,X_0)=0$. 且 $\forall f\in X^*$. $f(X_0)=0 \Rightarrow f(x)=0$.

若 $x\notin X_0$ $\therefore d=\rho(x,X_0)>0$.

对 $\forall f\in X^*$ $f(X_0)=0$ $\|f\|=1$.

对 $\forall \varepsilon>0$ 取 $y\in X_0$. $d(x,y)<d+\varepsilon$.

$\therefore |f(x)| = |f(y)+f(x-y)| = |f(x-y)| \leq \|f\|\|x-y\| < d+\varepsilon$.

令 $\varepsilon\to 0$ $|f(x)|\leq d$. $\Rightarrow$ RHS $\leq d$. for sup of f.

由定理 2.4.7. $\exists f\in X^*$. $f(X_0)=0$, $\|f\|=1$. $f(x)=d$. $\Rightarrow$ RHS $\geq d$. $\square$.

2.4.6. $\Rightarrow$: $\left|f\left(\sum_{k=1}^n a_k x_k\right)\right| = \left|\sum_{k=1}^n a_k C_k\right| \leq \|f\|\left\|\sum_{k=1}^n a_k x_k\right\| \leq M\left\|\sum_{k=1}^n a_k x_k\right\|$

$\Leftarrow$: 取 $X_0=\operatorname{span}\{x_1,\cdots x_n\}$ 令 $f: X_0\to \mathbb{K}$.

$$\sum_{k=1}^n a_k x_k \mapsto \sum_{k=1}^n a_k C_k$$

$\therefore$ 由条件 $\forall x\in X_0$ $|f(x)|\leq M\|x\| \Rightarrow \|f\|_{X_0}\leq M$ 且 $f(x_k)=C_k$

由 HBT, 延拓存在. $\square$

2.4.7. 令 $X_0 = \mathrm{span}\{x_2, \cdots, x_n\}$ 由 $x_1 \cdots x_n$ 线性无关 $\Rightarrow x_1 \notin X_0 \Rightarrow d(x_1, X_0) > 0$

$\therefore \exists f_1 \in X^*$, s.t. $f(X_0) = 0$. $f(x_1) = 1$. By 定理 2.4.7. 即 $f_1(x_i) = \delta_{i1}$ □.

补充: 证明: $A \subseteq X$ 的凸包为 A 中任意凸组合的全体

即 $\bigcap_{A \subseteq B \text{凸}} B = \left\{ \sum_{i=1}^{n} \lambda_i x_i \;\middle|\; \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in A \right\}$

pf: 记 RHS $= C$. 对 $x = \sum_{i=1}^{n} \lambda_i x_i$. $y = \sum_{j=1}^{m} \mu_j y_j \in C$.

有 $\lambda_i, \mu_j \geq 0$, $\sum_{i=1}^{n} \lambda_i = \sum_{j=1}^{m} \mu_j = 1$. $x_i, y_j \in A$

$\therefore$ 对 $\forall t \in [0,1]$ $tx + (1-t)y = \sum_{i=1}^{n} t\lambda_i x_i + \sum_{j=1}^{m} (1-t)\mu_j y_j \in C$

$\therefore$ C 为凸集 $\Rightarrow$ LHS $\subseteq C$.

另一方面, 对 $\forall B$. $B \supseteq A$. B 为凸集, 对 $\forall x_i \in A$, $i = 1, \cdots n$. $\forall \sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$

有 $\sum_{i=1}^{n} \lambda_i x_i \in C$ $\Rightarrow C \subseteq B$ $\Rightarrow$ $C \subseteq$ LHS

$\therefore$ LHS $= C$ □.