2.1.1. $\Rightarrow$ 由于 $T \in \mathcal{L}(X,Y)$, T 有界, 线性, 结论显然.

$\Leftarrow$ $T\{x \in X : \|x\| = 1\}$ 有界 $\therefore T$ 有界 $\square$.

2.1.2. (1) "$\geq$" for $\forall x$, $\|x\| \leq 1$.

$$\|Ax\| = \|x\| \cdot \|A \tfrac{x}{\|x\|}\| \leq \|A \tfrac{x}{\|x\|}\| \leq \sup_{\|x\|=1} \|Ax\| = \|A\|$$

take sup for x, $\|x\| \leq 1$

$$\sup_{\|x\| \leq 1} \|Ax\| \leq \|x\|$$

"$\leq$" $\|x\| = \sup_{\|x\|=1} \|Ax\| \leq \sup_{\|x\| \leq 1} \|Ax\|$.

(2) "$\geq$" $\sup_{\|x\|<1} \|Ax\| \leq \sup_{\|x\| \leq 1} \|Ax\| = \|A\|$.

"$\leq$" for $\forall \varepsilon > 0$, take x s.t. $\|x\| = 1$.

$\|Ax\| \geq \|A\| - \varepsilon$. (we only need to consider $A \neq 0$ otherwise, it is trivial.)

$$\therefore \|A\| - \varepsilon \leq \|Ax\| \leq \frac{1}{1-\varepsilon} \|A(1-\varepsilon)x\| \leq \frac{1}{1-\varepsilon} \sup_{\|x\|<1} \|Ax\|$$

let $\varepsilon \to 0$. $\|A\| \leq \sup_{\|x\|<1} \|Ax\|$ $\square$.

2.1.5. $f \neq 0$ $\therefore \exists x$ s.t. $f(x) \neq 0$ $\therefore f\left(\frac{x}{f(x)}\right) = 1 \Rightarrow d \neq +\infty$.

for $\forall x$, $\|x\| = 1$, $f(x) \neq 0$

we have $f\left(\frac{x}{f(x)}\right) = 1 \Rightarrow \left\|\frac{x}{f(x)}\right\| \geq d$. $\frac{|f(x)|}{\|x\|} \leq \frac{1}{d}$.

i.e. for $\forall \|x\| = 1$ $f(x) = 0$, or $\frac{|f(x)|}{\|x\|} \leq \frac{1}{d} \Rightarrow \|f\| \leq \frac{1}{d}$.

for $\forall \varepsilon > 0$, take x s.t. $\|x\| \leq d + \varepsilon$, $f(x) = 1$.

(since f is bounded, we know $d \neq 0$).

$\therefore \|f\| \geq \frac{|f(x)|}{\|x\|} \geq \frac{1}{d+\varepsilon}$. let $\varepsilon \to 0$ $\|f\| \geq \frac{1}{d}$ $\square$.

2.1.6. if $f = 0$, it is trivial.

if $f \neq 0$, for $\forall \delta > 0$, $\exists x_0$, $\|x_0\| = 1$, s.t. $|f(x_0)| \geq \|f\| - \delta$.

WLOG, we assume $f(x_0) \geq \|f\| - \delta$, otherwise we can consider $-x_0$.

$\therefore$ let $x_1 = \frac{\|f\|}{f(x_0)} x_0$ $\therefore f(x_1) = \|f\|$ $\|x_1\| = \frac{\|f\|}{f(x_0)} \leq \frac{\|f\|}{\|f\| - \delta} < 1 + \varepsilon$

if δ small enough. $\square$

2.1.7. (1) for $x, y \in N(T)$ $T(ax+by) = aTx + bTy = 0 \Rightarrow ax + by \in N(T)$

$\therefore N(T)$ is a linear subspace.

for $x_n \in N(T)$, $x_n \to x_0$. since T is continous $0 = Tx_n \to Tx_0$

$\therefore Tx_0 = 0 \Rightarrow x_0 \in N(T)$ $\therefore N(T)$ is closed.

(2) NO! Take any Banach space X. and its hamel basis $\{e_\lambda\}_{\lambda\in\Lambda}$.

we choose a sequence $\{e_n\}_{n=1}^{+\infty}$, then let $\{e_\lambda\}_{\lambda\in\Lambda} = \{e_n\}_{n=1}^{+\infty} \amalg \{f_\lambda\}_{\lambda\in\Lambda}$

$\therefore$ we define
$$
\begin{aligned}
T:\ & X \to X. \\
& e_n \mapsto n e_n \\
& f_\lambda \mapsto f_\lambda
\end{aligned}
$$
$\therefore f$ is linear. $N(T)=0$

but T is not bounded. since $\|T\| \geq \frac{\|Te_n\|}{\|e_n\|} = n$ for $\forall n$.

Rmk: zgq书的答案是不对的

因为 ℓ^1 空间在 ℓ^∞ 范数下不是完备的

例如 $a_n = (1, 2, \cdots, \frac{1}{n}, \cdots)$ 它是Cauchy列 但不收敛.

(3). $\Rightarrow$ By (1)

$\Leftarrow$ we suppose f is not bounded. $\therefore \exists x_n$, $\|x_n\|=1$. s.t. $|f(x_n)| \geq n$.

WLOG we assume $f(x_n) \geq n$. otherwise, we can consider $e^{i\theta} x_n$.

$\therefore$ we define $y_n = \frac{x_n}{f(x_n)}$ $\therefore f(y_n)=1$. $\|y_n\| \leq \frac{1}{n}$ $\therefore y_n \to 0$

$\therefore y_n - y_1 \in N(T)$. $\Rightarrow -y_1 \in N(T)$ since $N(T)$ is closed.

However, $f(-y_1) = -1 \neq 0$. Contradiction!

$\therefore f$ is bounded.

HW: 1°. $T: X \to Y$. linear. $\dim X = n < +\infty$.

we choose $\{e_i\}_{i=1}^n$ s.t. $X = \operatorname{span}\{e_1 \cdots e_n\}$

we define $\|x\|_0 = \left(\sum_{i=1}^n a_i^2\right)^{1/2}$ for $x = \sum_{i=1}^n a_i e_i$.

$\therefore \exists C > 1$. s.t. $\frac{1}{C}\|x\| \leq \|x\|_0 \leq C\|x\|$.

we set $M = \sup_{i=1,2,\cdots n} \|Te_i\| < +\infty$

$\therefore$ for $\forall x = \sum_{i=1}^n a_i e_i$ $\|x\|=1$ $\therefore \|x\|_0 = \left(\sum_{i=1}^n a_i^2\right)^{1/2} \leq C$.

$$\|Tx\| = \left\|\sum_{i=1}^n a_i Te_i\right\| \leq M \sum_{i=1}^n |a_i|. \leq M\sqrt{n\sum_{i=1}^n a_i^2} \leq M\sqrt{n}C < +\infty.$$

$\therefore T$ is bounded.

2°. like 2.1.7(2). take Hamel basis $\{e_n\}_{n=1}^{+\infty} \amalg \{f_\lambda\}_{\lambda\in\Lambda}$ of X

and define
$$
\begin{aligned}
T:\ & X \to Y. \\
& e_n \mapsto n y. \\
& f_\lambda \mapsto y.
\end{aligned}
$$
where $y \in Y$. $y \neq 0$. □