

第八周作业答案

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习题 10.3

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(8)

由对称性, 不妨设 $a \geq 0$, 则 $x \geq 0$ 。

令 $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$, 则

$$(x^2 + y^2 + z^2)^2 \leq a^3 x \implies r^3 \leq a^3 \sin \theta \cos \varphi \implies r \leq a \sqrt[3]{\sin \theta \cos \varphi}$$

于是体积为

$$\begin{aligned} \iiint_V dx dy dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^\pi d\theta \int_0^{a \sqrt[3]{\sin \theta \cos \varphi}} r^2 \sin \theta dr \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^\pi d\theta \int_0^{a \sqrt[3]{\sin \theta \cos \varphi}} r^2 \sin \theta dr \\ &= \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^\pi \sin^2 \theta \cos \varphi d\theta \\ &= \frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi d\varphi \int_0^\pi \sin^2 \theta d\theta \\ &= \frac{\pi a^3}{3} \end{aligned}$$

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注意到

$$x^2 + y^2 + z^2 \leq x + y + z \iff \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 \leq \frac{3}{4}$$

这是一个球 B , 其体积为 $\frac{\sqrt{3}\pi}{2}$ 。

令 $x = \frac{1}{2} + r \sin \theta \cos \varphi, y = \frac{1}{2} + r \sin \theta \sin \varphi, z = \frac{1}{2} + r \cos \theta$, 于是平均值为

$$\begin{aligned} & \frac{2}{\sqrt{3}\pi} \iiint_{x^2+y^2+z^2 \leq x+y+z} f(x, y, z) dx dy dz \\ &= \int_0^{\frac{\sqrt{3}}{2}} dr \int_0^\pi d\theta \int_0^{2\pi} r^2 \sin \theta \left(\frac{3}{4} + r^2 + \sin \theta \cos \varphi + \sin \theta \sin \varphi + \cos \theta \right) d\varphi \\ &= 2\pi \int_0^{\frac{\sqrt{3}}{2}} dr \int_0^\pi r^2 \sin \theta \left(\frac{3}{4} + r^2 + \cos \theta \right) d\theta \\ &= 4\pi \int_0^{\frac{\sqrt{3}}{2}} \left(r^4 + \frac{3}{4}r^2 \right) dr \\ &= \frac{3\sqrt{3}}{5}\pi \end{aligned}$$

两式相比, 比值为 $\frac{6}{5}$ 。

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$$2\pi k(R^2 - r^2)$$

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$$b = \frac{\sqrt{6}}{3}a$$

16

$$(0, 0, \frac{4}{5}a)$$

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$$\pi GR\rho \sin^2 \alpha$$

习题 10.4

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(1)

$$\int \cdots \int_{[0,1]^n} (x_1^2 + \cdots + x_n^2) dx_1 \cdots dx_n = \sum_{i=1}^n \int \cdots \int_{[0,1]^n} x_i^2 dx_1 \cdots dx_n = \sum_{i=1}^n \int_0^1 x_i^2 dx_i = \frac{n}{3}$$

(2)

$$\begin{aligned} & \int \cdots \int_{[0,1]^n} (x_1 + \cdots + x_n)^2 dx_1 \cdots dx_n \\ &= \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_n)^2 dx_n \\ &= \frac{1}{3} \int_0^1 dx_1 \cdots \int_0^1 ((x_1 + \cdots + x_{n-1} + 1)^3 - (x_1 + \cdots + x_{n-1})^3) dx_{n-1} \\ &= \frac{1}{3} \int_0^1 dx_1 \cdots \int_0^1 (3(x_1 + \cdots + x_{n-1})^2 + 3(x_1 + \cdots + x_{n-1}) + 1) dx_{n-1} \\ &= \frac{1}{3} + \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1}) dx_{n-1} \\ &= \frac{1}{3} + \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \sum_{i=1}^{n-1} \int_0^1 dx_1 \cdots \int_0^1 x_i dx_{n-1} \\ &= \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-1})^2 dx_{n-1} + \frac{n-1}{2} + \frac{1}{3} \\ &= \int_0^1 dx_1 \cdots \int_0^1 (x_1 + \cdots + x_{n-2})^2 dx_{n-2} + \frac{n-1}{2} + \frac{n-2}{2} + \frac{2}{3} \\ &= \cdots \\ &= \frac{n(n-1)}{4} + \frac{n}{3} = \frac{n(3n+1)}{12} \end{aligned}$$

(3)

$$\begin{aligned} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} x_1 x_2 \cdots x_n dx_n &= \frac{1}{2} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-2}} x_1 x_2 \cdots x_{n-2} x_{n-1}^3 dx_{n-1} \\ &= \frac{1}{8} \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-3}} x_1 x_2 \cdots x_{n-3} x_{n-2}^5 dx_{n-2} \\ &= \cdots \\ &= \frac{1}{(n-1)! 2^{n-1}} \int_0^1 x_1^{2^{n-1}} dx_1 \\ &= \frac{1}{n! 2^n} \end{aligned}$$

第 10 章综合习题

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(1)

由题

$$\begin{aligned} I_2 &= \int_0^1 \sin\left(\ln \frac{1}{x}\right) \frac{x^b - x^a}{\ln x} dx \\ &= \int_0^1 \sin\left(\ln \frac{1}{x}\right) \int_a^b x^y dy dx \\ &= \int_a^b dy \int_0^1 \sin\left(\ln \frac{1}{x}\right) x^y dx \end{aligned}$$

令 $t = -\ln x$, 则 $x = e^{-t}$, 进而 $dx = -e^{-t} dt$, 于是

$$\begin{aligned} \int_0^1 \sin\left(\ln \frac{1}{x}\right) x^y dx &= \int_0^{+\infty} e^{-t(y+1)} \sin t dt \\ &= \frac{1}{2i} \int_0^{+\infty} e^{-t(y+1)} (e^{it} - e^{-it}) dt \\ &= \left(-\frac{e^{-t(y+1-i)}}{2i(y+1-i)} + \frac{e^{-t(y+1+i)}}{2i(y+1+i)} \right) \Bigg|_{t=0}^{+\infty} \\ &= \frac{1}{2i(y+1-i)} - \frac{1}{2i(y+1+i)} \\ &= \frac{1}{1+(y+1)^2} \end{aligned}$$

因此

$$I_1 = \int_a^b \frac{1}{1+(y+1)^2} dy = \arctan(b+1) - \arctan(a+1)$$

(2)

由题

$$\begin{aligned} I_1 &= \int_0^1 \cos\left(\ln \frac{1}{x}\right) \frac{x^b - x^a}{\ln x} dx \\ &= \int_0^1 \cos\left(\ln \frac{1}{x}\right) \int_a^b x^y dy dx \\ &= \int_a^b dy \int_0^1 \cos\left(\ln \frac{1}{x}\right) x^y dx \end{aligned}$$

令 $t = -\ln x$, 则 $x = e^{-t}$, 进而 $dx = -e^{-t} dt$, 于是

$$\begin{aligned} \int_0^1 \cos\left(\ln \frac{1}{x}\right) x^y dx &= \int_0^{+\infty} e^{-t(y+1)} \sin t dt \\ &= \frac{1}{2} \int_0^{+\infty} e^{-t(y+1)} (e^{it} + e^{-it}) dt \\ &= -\left(\frac{e^{-t(y+1-i)}}{2(y+1-i)} + \frac{e^{-t(y+1+i)}}{2(y+1+i)}\right) \Big|_{t=0}^{+\infty} \\ &= \frac{1}{2(y+1-i)} + \frac{1}{2(y+1+i)} \\ &= \frac{y+1}{1+(y+1)^2} \end{aligned}$$

因此

$$I_2 = \int_a^b \frac{y+1}{1+(y+1)^2} dy = \frac{1}{2} \ln \frac{1+(b+1)^2}{1+(a+1)^2}$$

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不难得到

$$\iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy = \iint_{B(0,1)} \left(x^2 + y^2 - \frac{x+y}{\sqrt{2}} \right) dx dy + 2 \iint_{B(\frac{1}{2\sqrt{2}}, \frac{1}{2})} \left(\frac{x+y}{\sqrt{2}} - x^2 - y^2 \right) dx dy$$

一方面, 令 $x = r \cos \theta, y = r \sin \theta$, 则

$$\begin{aligned} I_1 &= \iint_{B(0,1)} \left(x^2 + y^2 - \frac{x+y}{\sqrt{2}} \right) = \int_0^{2\pi} d\theta \int_0^1 r \left(r^2 + \frac{r \cos \theta + r \sin \theta}{2} \right) dr \\ &= \int_0^{2\pi} \left(\frac{1}{4} + \frac{1}{3\sqrt{2}} (\cos \theta + \sin \theta) \right) d\theta = \frac{\pi}{2} \end{aligned}$$

另一方面, 令 $x = \frac{1}{2\sqrt{2}} + \frac{r}{2} \cos \theta, y = \frac{1}{2\sqrt{2}} + \frac{r}{2} \sin \theta$, 则

$$I_2 = \iint_{B(\frac{1}{2\sqrt{2}}, \frac{1}{2})} \left(\frac{x+y}{\sqrt{2}} - x^2 - y^2 \right) = \frac{1}{16} \int_0^{2\pi} d\theta \int_0^1 r(1-r^2) dr = \frac{\pi}{32}$$

因此

$$\iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy = I_1 - 2I_2 = \frac{9\pi}{16}$$

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设 $x = ar \cos \theta, y = ar \sin \theta$, 则

$$\begin{aligned} (x-a)^2 + (y-a)^2 \leq a^2 &\implies (r - \cos \theta - \sin \theta)^2 \leq \sin 2\theta \\ (x^2 + y^2)^2 \leq 8a^2 xy &\implies r^4 \leq 8r^2 \cos \theta \sin \theta \implies r^2 \leq 4 \sin 2\theta \end{aligned}$$

进一步

$$\sqrt{\sin 2\theta} \geq \sin \theta + \cos \theta - \sqrt{\sin 2\theta} \implies 8 \sin 2\theta \geq 1 \implies \frac{1}{2} \arcsin \frac{1}{8} \leq \theta \leq \frac{\pi}{2} - \frac{1}{2} \arcsin \frac{1}{8}$$

于是根据对称性, D 的面积为

$$\begin{aligned}
 \iint_D dx dy &= a^2 \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{1}{8}} d\theta \int_{\sin \theta + \cos \theta - \sqrt{\sin 2\theta}}^{2\sqrt{\sin 2\theta}} r dr \\
 &= \frac{a^2}{2} \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{1}{8}} \left(2 \sin 2\theta - 1 + 2(\sin \theta + \cos \theta) \sqrt{\sin 2\theta} \right) d\theta \\
 &= a^2 \cos \arcsin \frac{1}{8} - \frac{a^2}{2} \left(\frac{\pi}{2} - \arcsin \frac{1}{8} \right) + \frac{a^2}{2} \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{1}{8}} \left(2(\sin \theta + \cos \theta) \sqrt{\sin 2\theta} \right) d\theta \\
 &= \frac{3\sqrt{7}a^2}{8} - \frac{\pi a^2}{4} + \frac{a^2}{2} \arcsin \frac{1}{8} + \frac{a^2}{2} \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{4}} \left(2(\sin \theta + \cos \theta) \sqrt{\sin 2\theta} \right) d\theta
 \end{aligned}$$

令 $t = \theta + \frac{\pi}{4}$, 则

$$\begin{aligned}
 \int_{\frac{1}{2} \arcsin \frac{1}{8}}^{\frac{\pi}{4}} \left(2(\sin \theta + \cos \theta) \sqrt{\sin 2\theta} \right) d\theta &= \sqrt{2} \int_{\frac{1}{2} \arcsin \frac{1}{8} + \frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \sqrt{-\cos 2t} dt \\
 &= \sqrt{2} \int_{\frac{1}{2} \arcsin \frac{1}{8} + \frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \sqrt{1 - 2\cos^2 t} dt
 \end{aligned}$$

再令 $s = -\sqrt{2} \cos t$, 结合 $\arcsin \frac{1}{8} + \arccos \frac{1}{8} = \frac{\pi}{2}$ 知

$$\sqrt{2} \int_{\frac{1}{2} \arcsin \frac{1}{8} + \frac{\pi}{4}}^{\frac{\pi}{2}} \sin t \sqrt{1 - 2\cos^2 t} dt = \int_{\frac{\sqrt{7}}{4}}^0 \sqrt{1 - s^2} ds = -\frac{3\sqrt{7}}{32} - \frac{1}{2} \arcsin \frac{\sqrt{7}}{4}$$

综上, D 的面积为

$$\frac{21\sqrt{7}a^2}{64} - \frac{\pi a^2}{4} + \frac{a^2}{2} \arcsin \frac{1}{8} - \frac{a^2}{4} \arcsin \frac{\sqrt{7}}{4}$$

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设 $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$, 则

$$(x^2 + y^2)^2 + z^4 \leq y \implies r^4 \sin^4 \theta + r^4 \cos^4 \theta \leq r \sin \theta \sin \varphi \implies r^3 \leq \frac{\sin \theta \sin \varphi}{\sin^4 \theta + \cos^4 \theta}$$

于是体积为

$$\begin{aligned}
 \iiint_V dx dy dz &= \int_0^\pi d\varphi \int_0^\pi d\theta \int_0^{\sqrt[3]{\frac{\sin \theta \sin \varphi}{\sin^4 \theta + \cos^4 \theta}}} r^2 \sin \theta dr \\
 &= \frac{1}{3} \int_0^\pi \sin \varphi d\varphi \int_0^\pi \frac{\sin^2 \theta}{\sin^4 \theta + \cos^4 \theta} d\theta \\
 &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta}{\sin^4 \theta + \cos^4 \theta} d\theta \\
 &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta (\tan^2 \theta + 1)}{\tan^4 \theta + 1} d\theta \\
 &= \frac{\sqrt{2}\pi}{3}
 \end{aligned}$$

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证明. 换元

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

则不难得到

$$x^2 + y^2 \leq 1 \iff u^2 + v^2 \leq a^2 + b^2$$

于是

$$\begin{aligned} \iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy &= \iint_{u^2+v^2 \leq a^2+b^2} f(u+c) dx dy \\ &= \int_{-1}^1 du \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} f(u+c) dv \\ &= 2 \int_{-\sqrt{a^2+b^2}}^{\sqrt{a^2+b^2}} \sqrt{a^2+b^2-u^2} f(u+c) du \end{aligned}$$

最后, 令 $t = \frac{u}{\sqrt{a^2+b^2}}$, 得到

$$\iint_{x^2+y^2 \leq 1} f(ax+by+c) dx dy = 2 \int_{-1}^1 \sqrt{1-t^2} f(t\sqrt{a^2+b^2}+c) dt$$

□

附加题

1 用五种方法计算椭圆

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

的体积。

证明. 答案是 $\frac{4\pi}{3}abc$, 五种方法为: 先一后二、先二后一、球坐标换元、柱坐标换元、放缩。 □

2 计算积分

$$I_1 = \iiint_{x^2+y^2+z^2 \leq 1} \cos(ax+by+cz) dV \quad I_2 = \iiint_{x^2+y^2+z^2 \leq 1} (ax+by+cz)^m dV$$

证明. 记 $r^2 = a^2 + b^2 + c^2$. 旋转坐标系, 得到

$$\begin{aligned} I_1 &= \iiint_{x^2+y^2+z^2 \leq 1} \cos(ax+by+cz) dV = \iiint_{x^2+y^2+z^2 \leq 1} \cos(\sqrt{a^2+b^2+c^2}x) dV \\ &= \int_{-1}^1 dx \iint_{y^2+z^2 \leq 1-x^2} \cos(rx) dS = \pi \int_{-1}^1 (1-x^2) \cos(rx) dx \\ &= -\frac{4\pi}{\sqrt{a^2+b^2+c^2}} \cos \sqrt{a^2+b^2+c^2} + \frac{4\pi}{(a^2+b^2+c^2)^{\frac{3}{2}}} \sin \sqrt{a^2+b^2+c^2} \end{aligned}$$

类似地

$$\begin{aligned} I_2 &= \iiint_{x^2+y^2+z^2 \leq 1} (ax + by + cz)^m dV = r^m \iiint_{x^2+y^2+z^2 \leq 1} x^m dV \\ &= r^m \int_{-1}^1 (1-x^2)x^m dV = \frac{\pi(1-(-1)^{m+1})}{m+1} - \frac{1-(-1)^{m+3}}{m+3} \\ &= \frac{2\pi}{(m+1)(m+3)} (1-(-1)^{m+1}) \end{aligned}$$

□

3 计算半径为 a 的 n 维球的体积

证明. 答案为

$$V = \begin{cases} \frac{2^k \pi^{n-1}}{(2k-1)!!} a^{2k-1}, & n = 2k - 1 \\ \frac{\pi^k}{k!} a^{2k}, & n = 2k \end{cases}$$

证明点击查看, 偷个懒 (◊ ∨ ◊)。

□

4 化简积分

$$I = \int \cdots \int_{\Omega} f\left(\sum_{i=1}^6 a_i x_i\right) dx_1 \cdots dx_6$$

这里 Ω 是 \mathbb{R}^6 的单位球。

证明. 对于 $\mathbf{a} = (a_1, \dots, a_6)$, 记 $a = |\mathbf{a}|$, 则旋转坐标系可得

$$I = \int \cdots \int_{\Omega} f(ax_1) dx_1 \cdots dx_6 = \int_{-1}^1 m(B(x_1)) f(ax_1) dx_1 = \frac{8\pi^2}{15} \int_{-1}^1 (1-x^2)^{\frac{5}{2}} f(ax) dx$$

这里

$$B(x_1) = \{\mathbf{x}' = (x_2, x_3, x_4, x_5, x_6) \mid |\mathbf{x}'|^2 < 1 - x_1^2\}$$

□

问题反馈

- 把积分范围要想明白;
- 第 10 章综合习题中没做出来的题目, 看过答案后多想想。