# 第九周作业答案

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## 习题 11.1

### 1

#### (1)

直接求导得

$$\mathbf{r}'(t) = \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t\right) \Longrightarrow |\mathbf{r}'(t)| = \sqrt{3}e^t$$

从而

$$s = \int_0^{2\pi} |\mathbf{r}'(t)| \,\mathrm{d}t = \sqrt{3}\int_0^{2\pi} e^t \,\mathrm{d}t = \sqrt{3}\left(e^{2\pi} - 1\right)$$

#### (3)

直接求导得

$$x' = -a\sin t \qquad y' = a\cos t \qquad z' = -a\tan t$$

即

$$|\mathbf{r}'(t)| = |a|\sqrt{1+\tan^2 t} = \frac{|a|}{\cos t}$$

从而

$$s = \int_0^{\frac{\pi}{4}} |\mathbf{r}'(t)| \,\mathrm{d}t = |a|\int_0^{\frac{\pi}{4}} \frac{1}{\cos t} \,\mathrm{d}t = |a|\ln(1+\sqrt{2})$$

#### (4)

直接带代换得到

$$r(z) = (x, y, z) = \left(\frac{z^2}{2a}, \frac{4z^{\frac{3}{2}}}{3\sqrt{2a}}, z\right),\ z \in [0, 2a]$$

求导得

$$r'(z) = \left(\frac{z}{a}, \sqrt{\frac{2z}{a}}, 1\right)$$

从而

$$s = \int_0^{2a} |\mathbf{r}'(z)| \,\mathrm{d}z = \int_0^{2a} \sqrt{\frac{z^2}{a^2} + \frac{2z}{a} + 1} \,\mathrm{d}z = \int_0^{2a} \left(\frac{z}{a} + 1\right) \mathrm{d}z = 4a$$

**(5)**

绕 $x$ 轴旋转 $\frac{\pi}{4}$，令

$$\tilde{x} = x \qquad \tilde{y} = \frac{y+z}{\sqrt{2}} \qquad \tilde{z} = \frac{z-y}{\sqrt{2}}$$

则两个曲面在新坐标系下的方程分别为

$$2a\tilde{x} = \tilde{y}^2 \qquad 2\tilde{x}^2 = 3\tilde{y}\tilde{z}$$

于是交线为

$$\mathbf{r}(t) = \left(\frac{t^2}{2a}, t, \frac{t^3}{6a^2}\right)$$

求导得

$$\mathbf{r}'(t) = \left(\frac{t}{a}, 1, \frac{t^2}{2a^2}\right)$$

因此原点到 $\mathbf{r}(T)$ 点得曲线弧长为

$$s = \int_0^T \sqrt{\frac{t^2}{a^2} + 1 + \frac{t^4}{4a^4}}\,\mathrm{d}t = \int_0^T \left(\frac{t^2}{2a} + 1\right)\mathrm{d}t = \frac{T^3}{6a} + T = \tilde{z} + \tilde{y} = \sqrt{2}z$$

## 2

**(2)**

不难得到

$$\mathrm{d}s = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + a^2}\,\mathrm{d}t = \sqrt{2}|a|\,\mathrm{d}t$$

于是

$$\int_L \frac{z^2}{x^2+y^2}\,\mathrm{d}s = \sqrt{2}|a| \int_0^{2a} t^2\,\mathrm{d}t = \frac{8\sqrt{2}\pi^3}{3}|a|$$

**(3)**

$$\int_L (x+y)\,\mathrm{d}s = \int_0^1 x\,\mathrm{d}x + \int_0^1 y\,\mathrm{d}y + \sqrt{2} = 1 + \sqrt{2}$$

**(6)**

直接换元得到

$$\int_L e^{\sqrt{x^2+y^2}}\,\mathrm{d}s = 2\int_0^a e^r\,\mathrm{d}r + a\int_0^{\frac{\pi}{4}} e^a\,\mathrm{d}\varphi = 2e^a - 2 + \frac{\pi a}{4}e^a$$

**(8)**

由题

$$\mathbf{r}(t) = (t\cos t, t\sin t, t)$$

求导得

$$\mathbf{r}(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$$

于是

$$\int_L z\,\mathrm{d}s = \int_0^{t_0} t\sqrt{t^2+2}\,\mathrm{d}t = \frac{1}{2}\int_0^{t_0^2}\sqrt{u+2}\,\mathrm{d}u = \frac{1}{3}(t_0^2+2)^{\frac{3}{2}} - \frac{2}{3}\sqrt{2}$$

**(10)**

令 $x = a\cos t, y = a\sin t$，则

$$\int_L (x^2+y^2+z^2)^n\,\mathrm{d}s = \int_0^{2\pi} a^{2n}|a|\,\mathrm{d}t = 2\pi a^{2n}|a|$$

**(11)**

由对称性

$$\int_L x^2\,\mathrm{d}s = \frac{1}{3}\int_L (x^2+y^2+z^2)\,\mathrm{d}s = \frac{a^2}{3}\int_L \mathrm{d}s = \frac{2\pi}{3}a^2|a|$$

**(12)**

$$\int_L (xy+yz+zx)\,\mathrm{d}s = \frac{1}{2}\int_L \left((x+y+z)^2 - (x^2+y^2+z^2)\right)\mathrm{d}s = -\frac{1}{2}\int_L a^2\,\mathrm{d}s = -\pi a^3$$

## 3

不难得到密度函数为

$$\rho(\mathbf{r}) = \frac{2}{r^2}$$

于是

$$m(t_0) = \left|\int_0^{t_0}\rho(\mathbf{r})\,\mathrm{d}s\right| = \left|\int_0^{t_0}\rho\left(e^t\cos t, e^t\sin t, e^t\right)\sqrt{3}e^t\right| = \sqrt{3}\left|\int_0^{t_0} e^{-t}\,\mathrm{d}t\right| = \sqrt{3}|1-e^{-t_0}|$$

## 4

$$I_x = \left(\frac{a^2}{2}+\frac{h^2}{3}\right)\sqrt{4\pi^2a^2+h^2}$$
$$I_y = \left(\frac{a^2}{2}+\frac{h^2}{3}\right)\sqrt{4\pi^2a^2+h^2}$$
$$I_z = a^2\sqrt{4\pi^2a^2+h^2}$$

## 5

不妨设半圆弧方程为

$$L: x^2+y^2=a^2,\ y\geq 0$$

于是由对称性，引力沿 $y$ 轴正向，大小为

$$\int_L \frac{GM\rho}{a^2}\sin\theta\,\mathrm{d}s = \frac{GM\rho}{a}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta = \frac{2GM\rho}{a}$$

# 习题 11.2

## 1

### (1)

令 $x = r\cos\theta, y = r\sin\theta$，则 $z = r$，此时

$$x^2 + y^2 \le 2x \Longleftrightarrow r \le 2\cos\theta$$

此时，曲面被参数化为

$$\mathbf{r}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$$

求导得

$$\left.\begin{aligned}\mathbf{r}_\rho &= (\cos\theta, \sin\theta, 1)\\ \mathbf{r}_\theta &= (-\rho\sin\theta, \rho\cos\theta, 0)\end{aligned}\right\} \Longrightarrow \begin{cases}E = 2\\ F = 0\\ G = \rho^2\end{cases}$$

因此

$$S = \iint_\Sigma \sqrt{EG - F^2}\,\mathrm{d}S = \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\rho\,\mathrm{d}\rho = 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta = \sqrt{2}\pi$$

### (4)

由题，$a > 0$。根据对称性，不难得到两曲面得交线为

$$\begin{cases}x^2 + y^2 = 2a^2\\ z = a\end{cases}$$

对于球面 $\Sigma_1$，令 $x = \sqrt{3}a\sin\theta\cos\varphi, y = \sqrt{3}a\rho\sin\theta\sin\varphi, z = \sqrt{3}a\cos\theta$，则对于

$$\mathbf{r}(\theta,\varphi) = \left(\sqrt{3}a\sin\theta\cos\varphi, \sqrt{3}a\rho\sin\theta\sin\varphi, \sqrt{3}a\cos\theta\right)$$

求导可得

$$\left.\begin{aligned}\mathbf{r}_\theta &= \left(\sqrt{3}a\cos\theta\cos\varphi, \sqrt{3}a\cos\theta\sin\varphi, -\sqrt{3}a\sin\theta\right)\\ \mathbf{r}_\varphi &= \left(-\sqrt{3}a\sin\theta\sin\varphi, \sqrt{3}a\sin\theta\cos\varphi, 0\right)\end{aligned}\right\} \Longrightarrow \begin{cases}E = 3a^2\\ F = 0\\ G = 3a^2\sin^2\theta\end{cases}$$

于是

$$S_1 = \iint_{\Sigma_1}\sqrt{EG - F^2}\,\mathrm{d}S = 3a^2\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\arccos\frac{1}{\sqrt{3}}}\sin\theta\,\mathrm{d}\theta = 2\left(3 - \sqrt{3}\right)\pi a^2$$

对于抛物面 $\Sigma_2$，令 $x = \sqrt{2}a\rho\cos\theta, y = \sqrt{2}a\rho\sin\theta$，这里 $0 \le \rho \le 1$。此时 $z = a\rho^2$。对于

$$\mathbf{r}(\rho,\theta) = \left(\sqrt{2}a\rho\cos\theta, \sqrt{2}a\rho\sin\theta, a\rho^2\right)$$

求导可得

$$\left.\begin{aligned}\mathbf{r}_\rho &= \left(\sqrt{2}a\cos\theta, \sqrt{2}a\sin\theta, 2a\rho\right)\\ \mathbf{r}_\theta &= \left(-\sqrt{2}a\rho\sin\theta, \sqrt{2}a\rho\cos\theta, 0\right)\end{aligned}\right\} \Longrightarrow \begin{cases} E = 4a^2\rho^2 + 2a^2 \\ F = 0 \\ G = 2a^2\rho^2 \end{cases}$$

于是

$$S_2 = \iint_{\Sigma_2} \sqrt{EG - F^2}\,\mathrm{d}S = 2a^2 \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho\sqrt{2\rho^2+1}\,\mathrm{d}\rho = \frac{2}{3}\pi a^2 \left(3\sqrt{3}-1\right)$$

综上

$$S = S_1 + S_2 = \frac{16}{3}\pi a^2$$

**(5)**

设曲面为

$$\mathbf{r}(y,z) = \left(y^2 + \frac{z^2}{2}, y, z\right)$$

求导可得

$$\left.\begin{aligned}\mathbf{r}_y &= (2y, 1, 0)\\ \mathbf{r}_z &= (z, 0, 1)\end{aligned}\right\} \Longrightarrow \begin{cases} E = 4y^2 + 1 \\ F = 2yz \\ G = z^2 + 1 \end{cases}$$

于是

$$\begin{aligned} S &= \int_\Sigma \sqrt{EG - F^2}\,\mathrm{d}S = \int_{4y^2+z^2\le 1} \sqrt{4y^2+z^2+1} \\ &= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r\sqrt{r^2+1}\,\mathrm{d}r = \pi \int_0^1 \sqrt{t+1}\,\mathrm{d}t = \frac{(2\sqrt{2}-1)\pi}{3} \end{aligned}$$

**(6)**

令 $x = \rho\cos\theta, y = \rho\sin\theta$，则由 $z \ge 0$ 知 $z = \rho$。进一步

$$z \le \sqrt{2}\left(\frac{x}{2}+1\right) \Longleftrightarrow \sqrt{2}\rho \le \rho\cos\theta + 2 \Longleftrightarrow \rho \le \frac{2}{\sqrt{2}-\cos\theta}$$

此时对于

$$\mathbf{r}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$$

求导可得

$$\left.\begin{aligned}\mathbf{r}_\rho &= (\cos\theta, \sin\theta, 1)\\ \mathbf{r}_\theta &= (-\rho\sin\theta, \rho\cos\theta, 0)\end{aligned}\right\} \Longrightarrow \begin{cases} E = 2 \\ F = 0 \\ G = \rho^2 \end{cases}$$

于是

$$S = \iint_\Sigma \sqrt{EG - F^2}\,\mathrm{d}S = \sqrt{2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{2}{\sqrt{2}-\cos\theta}} \rho\,\mathrm{d}\rho = \sqrt{2}a \int_0^{2\pi} \frac{1}{2+\cos^2\theta - 2\sqrt{2}\cos\theta}\,\mathrm{d}\theta = 8\pi$$

**(7)**

直接对于

$$\mathbf{r}(\rho,\theta)=(\rho\cos\theta,\rho\sin\theta,h\theta)$$

求导可得

$$\left.\begin{aligned}\mathbf{r}_\rho&=(\cos\theta,\sin\theta,0)\\ \mathbf{r}_\theta&=(-\rho\sin\theta,\rho\cos\theta,h)\end{aligned}\right\}\implies\begin{cases}E=1\\ F=0\\ G=\rho^2+h^2\end{cases}$$

于是

$$S=\iint_\Sigma\sqrt{EG-F^2}\,\mathrm{d}S=\int_0^{2\pi}\mathrm{d}\theta\int_0^a\sqrt{\rho^2+h^2}\,\mathrm{d}\rho=2\pi\int_0^a\sqrt{\rho^2+h^2}\,\mathrm{d}\rho$$

令 $\rho=h\tan t$，则 $\mathrm{d}\rho=\frac{1}{\cos^2 t}\,\mathrm{d}t$，此时

$$S=2\pi\int_0^a\sqrt{\rho^2+h^2}\,\mathrm{d}\rho=2\pi h^2\int_0^{\arctan\frac{a}{h}}\frac{1}{\cos^3 t}\,\mathrm{d}t$$

再令 $s=\sin t$，则 $\mathrm{d}s=\cos t$，此时

$$\begin{aligned}S&=2\pi h^2\int_0^{\frac{a}{\sqrt{a^2+h^2}}}\frac{1}{(1-s)^2}\,\mathrm{d}s\\&=\frac{1}{2}\pi h^2\int_0^{\frac{a}{\sqrt{a^2+h^2}}}\left(\frac{1}{1+s}+\frac{1}{(1+s)^2}+\frac{1}{1-s}+\frac{1}{(1-s)^2}\right)\mathrm{d}s\\&=\frac{1}{2}\pi h^2\left(\ln\frac{\sqrt{a^2+h^2}+a}{\sqrt{a^2+h^2}-a}+\frac{2a}{h}\sqrt{a^2+h^2}\right)\end{aligned}$$

# 2

**(1)**

由对称性

$$\begin{aligned}\iint_S(x+y+z)\,\mathrm{d}S&=3\int_0^1\mathrm{d}x\int_0^1(x+y)\,\mathrm{d}y+3\int_0^1\mathrm{d}x\int_0^1(x+y+1)\,\mathrm{d}y\\&=6\int_0^1\mathrm{d}x\int_0^1(x+y)\,\mathrm{d}y+3\int_0^1\mathrm{d}x\int_0^1\mathrm{d}y\\&=6+3=9\end{aligned}$$

**(2)**

将该平面参数化为

$$\mathbf{r}(x,y)=(x,y,1-x-y)$$

其中 $0 \le x, y \le 1, x+y \le 1$。不难得到 $\sqrt{EG-F^2} = \sqrt{3}$，于是

$$\begin{aligned}\iint_S xyz\,\mathrm{d}S &= \sqrt{3}\int_0^1 \mathrm{d}x \int_0^{1-x} (xy - x^2y - xy^2)\,\mathrm{d}y \\ &= \frac{\sqrt{3}}{6}\int_0^1 x(1-x)^3\,\mathrm{d}x \\ &= \frac{\sqrt{3}}{6}\int_0^1 (1-x)^3\,\mathrm{d}x - \frac{\sqrt{3}}{6}\int_0^1 (1-x)^4\,\mathrm{d}x \\ &= \frac{\sqrt{3}}{120}\end{aligned}$$

**(3)**

不难得到交线为

$$\begin{cases} x^2+y^2=1 \\ z=1 \end{cases}$$

于是在锥面上进行参数化，得到

$$\mathbf{r}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho)$$

于是

$$\begin{aligned}\iint_S (x^2+y^2)\,\mathrm{d}S &= \iint_{S_1}(x^2+y^2)\,\mathrm{d}S + \iint_{S_2}(x^2+y^2)\,\mathrm{d}S \\ &= \int_0^{2\pi}\mathrm{d}\theta\int_{-1}^1 \rho^3\,\mathrm{d}\rho + \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 \rho^3\,\mathrm{d}\rho \\ &= \frac{\sqrt{2}+1}{2}\pi\end{aligned}$$

**(5)**

将曲面参数化为

$$\mathbf{r}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, r)$$

它满足

$$x^2+y^2 \le 2x \iff \rho \le 2\cos\theta$$

直接求导可得

$$\left.\begin{aligned}\mathbf{r}_\rho &= (\cos\theta, \sin\theta, 1) \\ \mathbf{r}_\theta &= (-\rho\sin\theta, \rho\cos\theta, 0)\end{aligned}\right\} \Longrightarrow \begin{cases} E=2 \\ F=0 \\ G=\rho^2 \end{cases}$$

于是

$$\begin{aligned}\iint_S (x^4-y^4+y^2z^2-x^2z^2+1)\,\mathrm{d}S &= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\cos\theta}\left(\rho^5(\cos^4\theta-\sin^4\theta+\sin^2\theta-\cos^2\theta)+\rho\right)\mathrm{d}\rho \\ &= 2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta = \sqrt{2}\pi\end{aligned}$$

**(6)**

将曲面参数化为

$$\mathbf{r}(\theta, h) = (R\cos\theta, R\sin\theta, h)$$

直接求导可得

$$\left.\begin{aligned} \mathbf{r}_\theta &= (-R\sin\theta, R\cos\theta, 0) \\ \mathbf{r}_h &= (0, 0, 1) \end{aligned}\right\} \Longrightarrow \begin{cases} E = R^2 \\ F = 0 \\ G = 1 \end{cases}$$

于是

$$\iint_S \frac{\mathrm{d}S^2}{r^2} = \int_0^{2\pi} \mathrm{d}\theta \int_0^H \frac{R\,\mathrm{d}h}{R^2 + h^2} = 2\pi \arctan\frac{H}{R}$$

**3**

**(1)**

$$\iint_S (x^2 + y^2)\,\mathrm{d}S = \frac{2}{3}\iint_S (x^2 + y^2 + z^2)\,\mathrm{d}S = \frac{8}{3}\pi R^4$$

**(2)**

$$\iint_S (x + y + z)\,\mathrm{d}S = 2\iint_S x\,\mathrm{d}S + \iint_S z\,\mathrm{d}S = \iint_S z\,\mathrm{d}S$$

而

$$\iint_S z\,\mathrm{d}S = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\frac{\pi}{2}} a^3 \sin\theta\cos\theta\,\mathrm{d}\theta = \pi a^3$$

**5**

曲面可参数化为

$$\mathbf{r}(\rho, \theta) = \left(\rho\cos\theta, \rho\sin\theta, \frac{1}{2}\rho^2\right)$$

直接求导可得

$$\left.\begin{aligned} \mathbf{r}_\rho &= (\cos\theta, \sin\theta, \rho) \\ \mathbf{r}_\theta &= (-\rho\sin\theta, \rho\cos\theta, 0) \end{aligned}\right\} \Longrightarrow \begin{cases} E = 1 + \rho^2 \\ F = 0 \\ G = \rho^2 \end{cases}$$

于是

$$m\iint_S \rho\,\mathrm{d}S = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{\sqrt{2}} \rho^3\sqrt{\rho^2 + 1}\,\mathrm{d}\rho = \frac{\pi}{4}\int_0^2 t\sqrt{t + 1} = \frac{4}{15} + \frac{8\sqrt{3}}{5}$$

## 问题反馈

- 如果题目没说明，那么参数 $a$ 未必恒正，有时结果中要加绝对值；
- 曲面参数化是，Jacobi 已经蕴含在 $\sqrt{EG-F^2}$ 中，不要再多乘 $\rho$ 或 $\rho^2\sin\theta$；
- 换坐标系的时候要保持思路清晰，想明白点之间的对应关系；
- 看明白题目中曲面的范围有没有限制，如 $z \geq 0$。