第七周作业答案

于俊骜

2024 年 4 月 21 日

习题 10.1

1

(4)

$$\int_a^b \mathrm{d}y \int_y^b f(x,y)\,\mathrm{d}x = \int_a^b \mathrm{d}x \int_a^x f(x,y)\,\mathrm{d}y$$

(6)

$$\int_0^1 \mathrm{d}y \int_{\frac{1}{2}}^1 f(x,y)\,\mathrm{d}x + \int_1^2 \mathrm{d}y \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)\,\mathrm{d}x = \int_{\frac{1}{2}}^1 \mathrm{d}x \int_0^{\frac{1}{x}} f(x,y)\,\mathrm{d}y$$

2

(6)

$$\iint_D \frac{\sin y}{y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{y^2}^y \frac{\sin y}{y}\,\mathrm{d}x = \int_0^1 (\sin y - y\sin y)\,\mathrm{d}y = 1 - \sin 1$$

(8)

记

$$D_1 = \left\{(x,y) \in D | x + y \le \frac{\pi}{2}\right\} \qquad D_2 = \left\{(x,y) \in D | x + y > \frac{\pi}{2}\right\}$$

则

$$\begin{aligned}\iint_D |\cos(x+y)| \,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_1} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_2} \cos(x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\frac{\pi}{4}} \mathrm{d}y \int_x^{\frac{\pi}{2}-x} \cos(x+y)\,\mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}x \int_{\frac{\pi}{2}-x}^{x} \cos(x+y)\,\mathrm{d}y \\ &= \int_0^{\frac{\pi}{4}} (1-\sin 2y)\,\mathrm{d}y - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2x - 1)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} (1-\sin 2y)\,\mathrm{d}y \\ &= \frac{\pi}{2} - 1\end{aligned}$$

3

(1)

$$\begin{aligned}\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^{1} \mathrm{d}y \int_{-1}^{1} (x^2+y^2)\,\mathrm{d}x \\ &= 4\int_0^1 \mathrm{d}y \int_0^1 (x^2+y^2)\,\mathrm{d}x \\ &= 4\int_0^1 \left(\frac{1}{3}+y^2\right)\mathrm{d}y \\ &= \frac{8}{3}\end{aligned}$$

(2)

令

$$D_1 = \{(x,y)\in D | y \ge 0\} \qquad D_2 = \{(x,y)\in D | y < 0\}$$

则

$$\begin{aligned}\iint_D \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y &= \iint_{D_1} \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{D_1} \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_1} \sin x \sin y\,\mathrm{d}x\,\mathrm{d}y \\ &= 0\end{aligned}$$

5

证明．由对称性

$$\int_0^a \mathrm{d}x \int_0^x f(x)f(y)\,\mathrm{d}y = \int_0^a \mathrm{d}y \int_0^y f(x)f(y)\,\mathrm{d}x$$

因此

$$\begin{aligned}\int_0^a \mathrm{d}x\int_0^x f(x)f(y)\,\mathrm{d}y &= \frac{1}{2}\int_0^a \mathrm{d}x\int_0^x f(x)f(y)\,\mathrm{d}y+\frac{1}{2}\int_0^a \mathrm{d}y\int_0^y f(x)f(y)\,\mathrm{d}x\\ &= \frac{1}{2}\iint_{D_1} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y+\iint_{D_2} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\iint_{[0,a]\times[0,a]} f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2}\int_0^a f(x)\,\mathrm{d}x\int_0^a f(y)\,\mathrm{d}y\\ &= \frac{1}{2}\left(\int_0^a f(x)\,\mathrm{d}x\right)^2\end{aligned}$$

其中

$$D_1=\{(x,y)|0\le x,y\le a,y\le x\}\qquad D_2=\{(x,y)|0\le x,y\le a,x<y\}$$

另一方面

$$\int_0^a \mathrm{d}x\int_0^x f(y)\,\mathrm{d}y=\int_0^a \mathrm{d}y\int_y^a f(y)\,\mathrm{d}x=\int_0^a (a-y)f(y)\,\mathrm{d}y=\int_0^a (a-x)f(x)\,\mathrm{d}x$$

□

6

$$\begin{aligned}\iint_D \frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_c^d \mathrm{d}y\int_a^b \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\mathrm{d}x\\ &= \int_c^d (f_y(b,y)-f_y(a,y))\,\mathrm{d}t\\ &= f(b,d)-f(b,c)-f(a,d)+f(a,c)\end{aligned}$$

7

证明. 由题，$\forall\varepsilon>0$，$\exists\delta>0$，当 $\sqrt{x^2+y^2}<\delta$ 时，有

$$|f(x,y)-f(0,0)|<\varepsilon$$

因此，只要 $r<\delta$，就有

$$\begin{aligned}\left|\frac{1}{\pi r^2}\iint_{B(\mathbf{0},r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y-f(0,0)\right| &= \left|\frac{1}{\pi r^2}\iint_{B(\mathbf{0},r)} (f(x,y)-f(0,0))\,\mathrm{d}x\,\mathrm{d}y\right|\\ &\le \frac{1}{\pi r^2}\iint_{B(\mathbf{0},r)} |f(x,y)-f(0,0)|\,\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{1}{\pi r^2}\iint_{B(\mathbf{0},r)} \varepsilon\,\mathrm{d}x\,\mathrm{d}y\\ &= \varepsilon\end{aligned}$$

即

$$\lim_{r\to 0}\frac{1}{\pi r^2}\iint_{B(\mathbf{0},r)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = f(0,0)$$

□

习题 10.2

1

(1)

令 $t=r^2$，则

$$\begin{aligned}\int_0^R \mathrm{d}x\int_0^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)\,\mathrm{d}t &= \iint_D \ln(1+x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^R r\ln(1+r^2)\,\mathrm{d}r\\ &= \frac{\pi}{4}\int_0^{R^2}\ln(1+t)\,\mathrm{d}t\\ &= \frac{\pi}{4}(1+R^2)\ln(1+R^2)-\frac{\pi}{4}R^2\end{aligned}$$

2

(1)

令 $x=r\cos\theta, y=r\sin\theta$，则

$$x^2+y^2\le x+y \Longrightarrow r\le\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

于是由 $r\ge 0$ 知 $-\frac{\pi}{4}\le\theta\le\frac{3\pi}{4}$。于是

$$\begin{aligned}\iint_D\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\theta\int_0^{\sin\theta+\cos\theta} r^2\,\mathrm{d}r\\ &= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\sin\theta+\cos\theta)^3\,\mathrm{d}\theta\\ &= \frac{1}{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin^3\left(\theta+\frac{\pi}{4}\right)\mathrm{d}\theta\\ &= \frac{1}{\sqrt{2}}\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta\\ &= \frac{8\sqrt{2}}{9}\end{aligned}$$

(2)

令 $x=ar\cos\theta, y=br\sin\theta$，则

$$0\le y\le x\Longrightarrow 0\le\tan\theta\le\frac{b}{a}\Longrightarrow 0\le\theta\le\arctan\frac{b}{a}$$

且

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & ar\cos\theta\end{vmatrix}=abr$$

于是不难得到

$$\iint_D\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\,\mathrm{d}x\,\mathrm{d}y=ab\int_0^{\arctan\frac{b}{a}}\mathrm{d}\theta\int_0^2 r^2\,\mathrm{d}r=\frac{8}{3}ab\arctan\frac{b}{a}$$

(3)

令 $s=xy,t=\frac{y}{x}$，则

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{1}{2\sqrt{st}} & -\frac{1}{2t}\sqrt{\frac{s}{t}}\\ \frac{1}{2}\sqrt{\frac{t}{s}} & \frac{1}{2}\sqrt{\frac{s}{t}}\end{vmatrix}=\frac{1}{2t}$$

于是不难得到

$$\iint_D(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y=\int_1^2\mathrm{d}s\int_1^2\frac{1}{2t}\left(\frac{s}{t}+st\right)\mathrm{d}t=\int_1^2\frac{3s}{4}\,\mathrm{d}s=\frac{9}{8}$$

(4)

令 $s=\frac{x^2}{y},t=\frac{y^2}{x}$，则

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{2}{3}\sqrt[3]{\frac{t}{s}} & \frac{1}{3}\sqrt[3]{\frac{s^2}{t^2}}\\ \frac{1}{3}\sqrt[3]{\frac{t^2}{s^2}} & \frac{2}{3}\sqrt[3]{\frac{s}{t}}\end{vmatrix}=\frac{1}{3}$$

于是不难得到

$$\iint_D\mathrm{d}x\,\mathrm{d}y=\int_n^m\mathrm{d}s\int_b^a\frac{1}{3}\,\mathrm{d}t=\frac{(a-b)(m-n)}{3}$$

(7)

令 $s=x+y,t=x-y$，则

$$\frac{\partial(x,y)}{\partial(s,t)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}$$

于是不难得到

$$\iint_D\frac{x^2-y^2}{x+y+3}\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}\int_{-1}^1\mathrm{d}s\int_{-1}^1\frac{st}{\sqrt{s+3}}\,\mathrm{d}t=\int_{-1}^1\frac{s}{\sqrt{s+3}}\,\mathrm{d}s\int_{-1}^1 t\,\mathrm{d}t=0$$

(9)

令 $x=r\cos\theta,y=r\sin\theta$，并取

$$D_1=\{(x,y)\in D|x,y\geq 0\}$$

则

$$\iint_D|xy|\,\mathrm{d}x\,\mathrm{d}y=4\iint_{D_1}|xy|\,\mathrm{d}x\,\mathrm{d}y=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^a r^3|\sin\theta\cos\theta|\,\mathrm{d}r=\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,\mathrm{d}\theta\int_0^a r^3\,\mathrm{d}r=\frac{1}{8}a^4$$

3

(1)

设该图形第一象限的部分为 D。
不难得到两曲线在第一象限交于 $(1,1)$ 和 $(\sqrt{2},\frac{1}{\sqrt{2}})$，于是

$$S=2\iint_D \mathrm{d}x\,\mathrm{d}y=\int_1^{\sqrt{2}}\mathrm{d}x\int_{\frac{1}{x}}^{\sqrt{\frac{3-x^2}{2}}}\mathrm{d}y=\int_1^{\sqrt{2}}\left(\sqrt{\frac{3-x^2}{2}}-\frac{1}{x}\right)\mathrm{d}x=\frac{3}{\sqrt{2}}\arcsin\frac{1}{3}-\ln 2$$

(3)

令 $s=x+y,t=\frac{y}{x}$，则

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{1}{1+t} & -\frac{s}{(1+t)^2}\\ \frac{t}{1+t} & \frac{s}{(1+t)^2}\end{vmatrix}=\frac{s}{(1+t)^2}$$

于是

$$S=\iint_D \mathrm{d}x\,\mathrm{d}y=\int_a^b\mathrm{d}s\int_k^m\frac{s}{(1+t)^3}\,\mathrm{d}t=\int_a^b s\,\mathrm{d}s\int_k^m\frac{2}{(1+t)^3}\,\mathrm{d}t=\frac{b^2-a^2}{2}\left(\frac{1}{1+k}-\frac{1}{1+m}\right)$$

5

证明. 由对称性知

$$\int_0^1 e^{f(x)}\,\mathrm{d}x\int_0^1 e^{-f(y)}\,\mathrm{d}y=\iint_{[0,1]^2}e^{f(x)-f(y)}\,\mathrm{d}x\,\mathrm{d}y=\iint_{[0,1]^2}e^{f(y)-f(x)}\,\mathrm{d}x\,\mathrm{d}y$$

于是

$$\begin{aligned}\int_0^1 e^{f(x)}\,\mathrm{d}x\int_0^1 e^{-f(y)}\,\mathrm{d}y&=\frac{1}{2}\left(\iint_{[0,1]^2}e^{f(x)-f(y)}\,\mathrm{d}x\,\mathrm{d}y-\iint_{[0,1]^2}e^{f(y)-f(x)}\,\mathrm{d}x\,\mathrm{d}y\right)\\&=\frac{1}{2}\left(\iint_{[0,1]^2}\left(e^{f(x)-f(y)}+e^{f(y)-f(x)}\right)\mathrm{d}x\,\mathrm{d}y\right)\\&\geq\frac{1}{2}\left(\iint_{[0,1]^2}2\,\mathrm{d}x\,\mathrm{d}y\right)\\&=1\end{aligned}$$

□

习题 10.3

1

(1)

$$\iiint_V xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\int_1^2 x\,\mathrm{d}x\int_{-2}^1 y\,\mathrm{d}y\int_0^{\frac{1}{2}}\mathrm{d}z=-\frac{9}{8}$$

(2)

$$
\begin{aligned}
\iiint_V xy^2z^3\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\int_0^{xy} xy^2z^3\,\mathrm{d}z \\
&= \frac{1}{4}\int_0^1 \mathrm{d}x\int_0^x x^5y^6\,\mathrm{d}y \\
&= \frac{1}{28}\int_0^1 x^{12}\,\mathrm{d}x \\
&= \frac{1}{364}
\end{aligned}
$$

2

(1)

令 $x = r\cos\theta, y = r\sin\theta$，则

$$
y \le \sqrt{2x - x^2} \Longrightarrow x^2 + y^2 - 2x \le 0 \Longrightarrow r \le 2\cos\theta
$$

于是

$$
\begin{aligned}
\int_0^2 \mathrm{d}x\int_0^{\sqrt{2x-x^2}} \mathrm{d}y\int_0^a z\sqrt{x^2+y^2}\,\mathrm{d}z &= \int_0^a \mathrm{d}z\iint_{(x-1)^2+y^2\le 1} z\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^a z\,\mathrm{d}z\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta\int_0^{2\cos\theta} r^2\,\mathrm{d}x \\
&= \frac{2}{3}a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\,\mathrm{d}\theta \\
&= \frac{8}{9}a^2
\end{aligned}
$$

(2)

令 $x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$，于是

$$
\begin{aligned}
\int_{-R}^{R} \mathrm{d}x\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \mathrm{d}y\int_0^{\sqrt{R^2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,\mathrm{d}z &= \int_0^{2\pi} \mathrm{d}\varphi\int_0^{\frac{\pi}{2}} \mathrm{d}\theta\int_0^R r^4\sin^2\theta\,\mathrm{d}r \\
&= 2\pi\int_0^{\frac{\pi}{2}} \sin^2\theta\int_0^R r^4\,\mathrm{d}r \\
&= \frac{4\pi^2R^5}{15}
\end{aligned}
$$

(3)

令 $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$，于是

$$\begin{aligned}\int_0^1 \mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_0^{\sqrt{1-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,\mathrm{d}z &= \int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r^3\sin\theta\,\mathrm{d}r\\ &= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin\theta\int_0^R r^3\,\mathrm{d}r\\ &= \frac{\pi}{8}\end{aligned}$$

3

(1)

令 $x=r\cos\theta, y=r\sin\theta$，于是

$$\iiint_V (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^2\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_{\frac{r^2}{2}}^2 r^3\,\mathrm{d}z = \pi\int_0^2(2r^3-r^5) = \frac{16\pi}{3}$$

(3)

令 $x=r\cos\theta, y=r\sin\theta$，则

$$\sqrt{4-x^2-y^2}\le z\le\frac{x^2+y^2}{3}\Longrightarrow\sqrt{4-r^2}\le z\le\frac{r^2}{3}$$

于是

$$\iiint_V z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{3}}\mathrm{d}r\int_{\sqrt{4-r^2}}^{\frac{r^2}{3}} zr\,\mathrm{d}z = \pi\int_0^{\sqrt{3}} r\left(\frac{r^4}{9}-4+r^2\right)\mathrm{d}r = \frac{13\pi}{4}$$

(6)

令 $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$，于是

$$\begin{aligned}\iiint_V |x^2+y^2+z^2-1|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^2|r^2-1|r^2\sin\theta\,\mathrm{d}r\\ &= 4\pi\left(\int_0^1(r^2-r^4)\,\mathrm{d}r+\int_1^2(r^4-r^2)\,\mathrm{d}r\right)\\ &= 16\pi\end{aligned}$$

7

令 $x=r\sin\theta\cos\varphi, y=r\sin\theta\sin\varphi, z=r\cos\theta$，于是

$$F(t)=\iiint_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\mathrm{d}\theta\int_0^t f(r^2)r^2\sin\theta\,\mathrm{d}r = 4\pi\int_0^t f(r^2)r^2\,\mathrm{d}r$$

因此

$$F'(t)=4\pi\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(r^2)r^2\,\mathrm{d}r = 4\pi t^2 f(t^2)$$

8

令 $x = r\cos\theta, y = r\sin\theta$，于是

$$\iiint_{x^2+y^2+z^2\le 1} f(z)\,\mathrm{d}V = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\sqrt{1-z^2}}\int_0^1 f(z)r\,\mathrm{d}r = \pi\int_{-1}^1 f(z)(1-z^2)\,\mathrm{d}z$$

问题反馈

- 刚开始学累次积分换序时，可以多画画图，想象被积区域的形状，把上下限写对；
- 上课没讲过的函数，如 $\sinh^{-1}$，尽量不要用；
- 目前所学知识，可积性的等价判定只有分割求和取极限。