# Solutions to Homework 04

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## Folland. *Real Analysis*

### Exercise 1.5.25

*Proof.* We only need to consider the case $\mu(E) = +\infty$. Let

$$E_n = [n, n+1) \cap E,\ n \in \mathbb{Z}$$

thus $E_n$ is a measurable and $\mu(E_n) \leq 1 < +\infty$. By previous conclusion, for any $k > 0$, there exists a $F_\sigma$ set $K_{k,n} \subset E_n$ such that

$$\mu(E_n \backslash K_{k,n}) \leq \frac{1}{2^{|n|+k}}$$

Take union, we obtain $K_k \subset E$ for $F_\sigma$ set

$$K_k = \bigcup_{n=-\infty}^{\infty} K_{k,n}$$

and

$$\mu(E \backslash K_k) \leq \frac{3}{2^k}.$$

Therefore, the $F_\sigma$ set

$$K = \bigcup_{k=1}^{\infty} K_n \supset E$$

satisfies $\mu(E \backslash K) = 0$. We have proved a measurable set can be interiorly approximated by an $F_\sigma$ set.

Applying this conclusion, we construct an $F_\sigma$ set $K$ such that $K \subset E^c$ and $\mu(E^c \backslash K) = 0$. Hence $U = K^c$ is a $G_\delta$ set including $E$, such that $\mu(U \backslash E) = 0$. □

## Exercise 1.5.26

*Proof.* For a measurable set $E \subset \mathbb{R}$, we have

$$\mu(E) = \inf \left\{ \sum_{k=1}^{\infty} \mu(I_k) \middle| I_k = (a_k, b_k), E \subset \bigcup_{k=1}^{\infty} I_k \right\}.$$

Hence for any $\varepsilon > 0$, there is a sequence of open interval $\{I_k\}$ whose union covers $E$ such that

$$\sum_{k=1}^{\infty} \mu(I_k) - \frac{\varepsilon}{2} \leq \mu(E) \leq \sum_{k=1}^{\infty} \mu(I_k).$$

On the other hand, the convergence of the series above implies $\exists\, N > 0$, such that

$$0 < \sum_{k=N+1}^{\infty} \mu(I_k) < \frac{\varepsilon}{2}.$$

For a finite union of open intervals

$$A = \bigcup_{k=1}^{N} I_k,$$

we have

$$\mu(E \triangle A) = \mu(E \backslash A) + \mu(A \backslash E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

## Exercise 2.1.2

### (1)

*Proof.* First, we need a conclusion that $f : X \to \overline{\mathbb{R}}$ is measurable if $f$ is measurable on $f^{-1}(\mathbb{R})$ and $f^{-1}(\pm\infty) \in \mathcal{M}$.

For $E \in \mathcal{B}_{\overline{\mathbb{R}}}$, $f^{-1}(E)$ is measurable if $E \in \mathcal{M}_{\mathbb{R}}$. Otherwise, we without loss of generality assume $E = A \cup \{+\infty\}$ where $A \in \mathcal{B}_{\mathbb{R}}$, then

$$f^{-1}(E) = f^{-1}(A) \cup f^{-1}\{+\infty\} \in \mathcal{M}.$$

Back to the original problem, we have

$$\begin{aligned}(fg)^{-1}\{+\infty\} = &\left(f^{-1}\{+\infty\} \cap g^{-1}(0, +\infty]\right) \cup \left(f^{-1}(0, +\infty] \cap g^{-1}\{+\infty\}\right) \\ &\cup \left(f^{-1}\{-\infty\} \cap g^{-1}[-\infty, 0)\right) \cup \left(f^{-1}[-\infty, 0) \cap g^{-1}\{-\infty\}\right) \in \mathcal{M}.\end{aligned}$$

Similarly, we have $(fg)^{-1}\{-\infty\} \in \mathcal{M}$. As we know, $fg$ is measurable on $(fg)^{-1}(\mathbb{R})$ since $f$ and $g$ are respectively measurable on $f^{-1}(\mathbb{R})$ and $g^{-1}(\mathbb{R})$. According to the conclusion above, $fg$ is measurable on $X$. □

**(2)**

*Proof.* If $a = \pm\infty$, we without loss of generality assume $a = +\infty$. In that case, $f + g$ is obviously measurable on $h^{-1}(\mathbb{R})$. On the other hand, we have

$$\begin{aligned} h^{-1}\{+\infty\} &= \big(f^{-1}\{+\infty\} \cap g^{-1}(\mathbb{R})\big) \cup \big(f^{-1}(\mathbb{R}) \cap g^{-1}\{+\infty\}\big) \\ &\quad \cup \big(f^{-1}\{+\infty\} \cap g^{-1}\{-\infty\}\big) \cup \big(f^{-1}\{-\infty\} \cap g^{-1}\{+\infty\}\big) \in \mathcal{M}, \\ h^{-1}\{-\infty\} &= \big(f^{-1}\{-\infty\} \cap g^{-1}(\mathbb{R})\big) \cup \big(f^{-1}(\mathbb{R}) \cap g^{-1}\{-\infty\}\big) \in \mathcal{M}. \end{aligned}$$

If $a \in \mathbb{R}$, we can similarly show that $h^{-1}\{\pm\infty\} \in \mathcal{M}$. Let $E \in \mathbb{R}$ excluding $a$, then $h^{-1}(E)$ is obviously measurable. Otherwise, we have

$$\begin{aligned} h^{-1}(E) &= (f+g)^{-1}(E \backslash \{a\}) \cup (f+g)^{-1}\{a\} \\ &\quad \cup \big(f^{-1}\{+\infty\} \cap g^{-1}\{-\infty\}\big) \cup \big(f^{-1}\{-\infty\} \cap g^{-1}\{+\infty\}\big) \in \mathcal{M}. \end{aligned}$$

In summary, $h$ is always measurable. □

## Exercise 2.1.5

*Proof.* Fix an arbitrary Borel subset $E \subset \mathbb{R}$. If $f$ is measurable on both $A$ and $B$, then

$$f^{-1}(E) \cap (A \cup B) = (f^{-1}(E) \cap A) \cup (f^{-1}(E) \cap B) \in \mathcal{M}.$$

If $f$ is measurable on both $A \cup B$, then

$$f^{-1}(E) \cap A = f^{-1}(E) \cap (A \cup B) \cap A \in \mathcal{M}$$

since $A \in \mathcal{M}$. Similarly, we have $B \in \mathcal{M}$. □

## Exercise 2.3.21

*Proof.* $\Longrightarrow$:

It is obvious that

$$\left| \int |f_n| - \int |f| \right| \leq \int |f_n - f| \to 0.$$

$\Longleftarrow$:

Let $\{g_n\} = \{| f_n | + | f |\}$ be a sequence of $L^1$ functions that converge to $2|f|$ in $L^1$. The triangular inequality implies $| f_n - f | \leq g_n$, thus by generalized dominant convergence theorem, we have

$$\lim_{n\to\infty} \int | f_n - f | = \int \lim_{n\to\infty} | f_n - f | = 0$$

since $f_n \to f$ almost everywhere. □

## Exercise 2.5.49

*Proof.* For a null set $E \in \mathcal{M} \times \mathcal{N}$, we have

$$0 = \mu \times \nu(E) = \int \nu(E_x) \, \mathrm{d}\mu(x) = \int \mu(E^y) \, \mathrm{d}\nu(y),$$

which implies $\nu(E_x) = \mu(E^y) = 0$ for almost every $x$ and $y$.

For the $\lambda$-null set

$$A = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \neq 0\},$$

there exists an $E \in \mathcal{M} \times \mathcal{N}$ including $E_0$ such that $\mu \times \nu(E) = 0$. Therefore,

$$\int |f_x| \, \mathrm{d}\nu(y) = \int \chi_{A_x} |f_x| \, \mathrm{d}\nu(y) = 0,$$
$$\int |f^y| \, \mathrm{d}\mu(x) = \int \chi_{A^y} |f^y| \, \mathrm{d}\mu(x) = 0.$$

Back to the proof of the theorem 2.39, suppose $f \in L^+(\lambda) \cup L^1(\lambda)$. According to Proposition 2.12, there exists a $\mu \times \nu$-measurable function $g$ such that $f = g$, $\lambda$-almost everywhere. Since $g_x$ is $\nu$-measurable and $g^y$ is $\mu$-measurable.

Define a function $h = f - g$ that equals 0 almost everywhere, then the latter lemma implies that $h_x$ is $\nu$-measurable for almost every $x$ while $h^y$ is $\mu$-measurable for almost every $y$.

Particularly when $f \in L^1(\lambda)$, we have $h_x \in L^1(\nu)$ for almost every $x$. By Fubini's theorem, $g_x \in L^1(\nu)$ for almost every $x$, thus $f_x \in L^1(\nu)$ for almost every $x$. The counterpart for $f^y$ is still correct. Moreover, we have almost everywhere

$$\int h_x \, \mathrm{d}\nu(y) = 0 \Longrightarrow \int g_x \, \mathrm{d}\nu(y) = \int f_x \, \mathrm{d}\nu(y),$$
$$\int h^y \, \mathrm{d}\mu(x) = 0 \Longrightarrow \int g^y \, \mathrm{d}\mu(x) = \int f^y \, \mathrm{d}\nu(x).$$

If $f \in L^+(\lambda)$, then $g \in L^+(\mu \times \nu)$, thus Tonelli's theorem indicates functions

$$x \to \int g_x \, \mathrm{d}\nu(y) = \int f_x \, \mathrm{d}\nu(y),$$
$$y \to \int g^y \, \mathrm{d}\mu(x) = \int f^y \, \mathrm{d}\nu(x)$$

are both measurable. Correspondingly, if $f \in L^1(\lambda)$, then $g \in L^1(\mu \times \nu)$, thus Fubini's theorem indicates the two functions above are both integrable. An ultimately application of Fubini-Tonelli's theorem leads to the identity

$$\int f \, \mathrm{d}\lambda = \int \left( \int f \, \mathrm{d}\mu \right) \mathrm{d}\nu = \int \left( \int f \, \mathrm{d}\nu \right) \mathrm{d}\mu.$$

□