Solutions to Homework 14

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Folland. *Real Analysis*

Exercise 9.2.27

(1)

Proof. Abstract

$$h_\alpha(x) = \frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}|x|^{-\alpha},$$

which belongs to $L^1 + L^2$ when $\frac{n}{2} < \alpha < n$. Therefore, $\hat{h}_\alpha$ is well defined. By the definition of Gamma function, we have

$$\begin{aligned}
\int h_\alpha(x)e^{-\pi|x|^2}\,\mathrm{d}x &= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{\alpha}{2}}}\int_0^{+\infty} r^{n-\alpha-1}e^{-\pi r^2}\,\mathrm{d}r \\
&= \frac{\Gamma(\frac{\alpha}{2})|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\int_0^{+\infty} t^{\frac{n-\alpha-2}{2}}e^{-t}\,\mathrm{d}t \\
&= \frac{|\mathbb{S}^{n-1}|}{\pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right) \\
&= \int h_{n-\alpha}(x)e^{-\pi|x|^2}\,\mathrm{d}x.
\end{aligned}$$

For

$$\alpha \in \{z \in \mathbb{C} \mid 0 < \mathrm{Re} < n\},$$

it is easy to verify

$$\frac{\Gamma(\frac{\alpha}{2})}{\pi^{\frac{\alpha}{2}}}\int |x|^{-\alpha}e^{-\pi|x|^2}\,\mathrm{d}x$$

and

$$\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n-\alpha}{2}}}\int |x|^{\alpha-n}e^{-\pi|x|^2}\,\mathrm{d}x$$

are holomorphic functions with respect to α that coincide on $[\frac{n}{2}, n] \subset \mathbb{R}$. By the uniqueness theorem, they coincide everywhere in the domain of definition. $\square$

(2)

Proof. Let φ be an arbitrary Schwarz function, then

$$\begin{aligned}|\langle R_\alpha, \varphi\rangle| &\leq \left|\frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\right| \int |x|^{a-n}|\varphi(x)|\,\mathrm{d}x \\ &\leq C\|\varphi\|_\infty \int_{[-1,1]} |x|^{a-n} + C\int_{[-1,1]^c} \frac{1}{|x|^{2n-a}}|x|^n|\varphi(x)|\,\mathrm{d}x \\ &\leq C\|\varphi\|_\infty + C\|x^n\varphi\|_\infty.\end{aligned}$$

Therefore, R_α is a tempered distribution. Note that (1) implies

$$\langle \hat{R}_\alpha, \varphi\rangle = \langle R_\alpha, \hat{\varphi}\rangle = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\langle |x|^{\alpha-n}, \hat{\varphi}\rangle = \frac{1}{(2\pi)^\alpha}\langle |\xi|^{-\alpha}, \varphi\rangle,$$

hence

$$\hat{R}_\alpha = (2\pi|\xi|)^{-\alpha}\,.$$

$\square$

(3)

Proof. Let φ be an arbitrary Schwarz function, then

$$\int -\Delta R_2 \varphi\,\mathrm{d}x = \int (-\Delta R_2)^\wedge\,\check{\varphi}\,\mathrm{d}\xi = \int 4\pi^2|\xi|^2\hat{R}_2\check{\varphi}\,\mathrm{d}\xi = \int \check{\varphi}\,\mathrm{d}\xi = (\check{\varphi})^\wedge(0) = \varphi(0).$$

which implies

$$\Delta R_2 = -\delta.$$

$\square$

Exercise 9.3.30

Proof. It is obvious that the proposition is correct for $\alpha = 0$ and $\alpha = e_k$, where $1 \le k \le n$.

Assume

$$\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| \le C_\alpha \left(1+|\xi|\right)^{s-|\alpha|}$$

for every α such that $|\alpha| \le m-1$. For $|\alpha| = m \ge 2$, without loss of generality assume $\alpha_1 > 0$ and $\alpha^{(i)} = \alpha - ie_1$.

If $\alpha_1 = 1$, then

$$\begin{aligned}
\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &= s\left|\partial^{\alpha^{(1)}}\xi_1\left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&= s\left|\xi_1\partial^{\alpha^{(1)}}\left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&\le s\sqrt{1+|\xi|^2}\left|\partial^{\alpha^{(1)}}\left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&\le sC_{\alpha^{(1)}}\left(1+|\xi|\right)^{s-|\alpha|} \\
&= C_\alpha\left(1+|\xi|\right)^{s-|\alpha|}.
\end{aligned}$$

If $\alpha_1 \ge 2$, then

$$\begin{aligned}
\left|\partial^\alpha \left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &= s\left|\partial^{\alpha^{(1)}}\xi_1\left(1+|\xi|^2\right)^{\frac{s}{2}-1}\right| \\
&= s\left|\partial^{\alpha^{(2)}}\left(1+|\xi|^2\right)^{\frac{s}{2}-1} + (s-2)\partial^{\alpha^{(2)}}\xi_1^2\left(1+|\xi|^2\right)^{\frac{s}{2}-2}\right| \\
&\le sC_{\alpha^{(2)}}\left(1+|\xi|^2\right)^{s-2-|\alpha^{(2)}|} + s(s-2)\left|\partial^{\alpha^{(2)}}\xi_1^2\left(1+|\xi|^2\right)^{\frac{s}{2}-2}\right| \\
&\le \cdots\cdots \\
&\le C''_\alpha\sum_{j=1}^{[\frac{\alpha_1}{2}]-1}\left|\partial^{\alpha^{(2j)}}\left(1+|\xi|^2\right)^{\frac{s}{2}-2j}\right| \\
&+ \prod_{i=0}^{[\frac{\alpha_1}{2}]}(s-2i)\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{\frac{s}{2}-2j-2}\right| \\
&\le C''_\alpha\sum_{j=1}^{[\frac{\alpha_1}{2}]-1}\left(1+|\xi|^2\right)^{\frac{s}{2}-2j-|\alpha^{(2j)}|} \\
&+ \prod_{i=0}^{[\frac{\alpha_1}{2}]}(s-2i)\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\
&\le C'_\alpha\left(1+|\xi|^2\right)^{s-\alpha} + C'_\alpha\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right|.
\end{aligned}$$

For even α_1, we have

$$\begin{aligned}\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right| &= \left|\xi_1^{2j+2}\partial^{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\ &= \left|\left(1+|\xi|^2\right)^{j+1}\partial^{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\ &\le C_{\alpha^{(2j+2)}}\left(1+|\xi|^2\right)^{s-|\alpha|};\end{aligned}$$

while for odd α_1, the same inequality is derived from the case $\alpha_1 = 1$.

In summary, we obtain

$$\begin{aligned}\left|\partial^{\alpha}\left(1+|\xi|^2\right)^{\frac{s}{2}}\right| &\le C'_\alpha\left(1+|\xi|^2\right)^{s-\alpha} + C'_\alpha\left|\partial^{\alpha^{(2j+2)}}\xi_1^{2j+2}\left(1+|\xi|^2\right)^{s-2j-2}\right| \\ &\le C_\alpha\left(1+|\xi|^2\right)^{s-|\alpha|}.\end{aligned}$$

□

Exercise 9.3.36

Proof. Note that

$$\begin{aligned}\|\varphi_j\|_{H^s} &= \left(\int(1+|\xi|^2)^s\left|\int\varphi(x-a_j)e^{-2\pi ix\xi}\,\mathrm{d}x\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &= \left(\int(1+|\xi|^2)^s\left|\int\varphi(x)e^{-2\pi i(x+a_j)\xi}\,\mathrm{d}x\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &= \left(\int(1+|\xi|^2)^s\left|e^{2\pi ia_j\xi}\hat{\varphi}(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &= \left(\int(1+|\xi|^2)^s\left|\varphi(\xi)\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}} \\ &= \|\varphi\|_{H^s} < +\infty.\end{aligned}$$

This bound is independent of j.

Assume there is a convergent subsequence. Without loss of generality suppose $\{\varphi_j\}_{j=1}^{\infty}$ itself converges to φ_0 in H^s. By definition,

$$\|\varphi_i-\varphi_j\|_{H^s}^2 = \|\varphi_i\|_{H^s}^2 + \|\varphi_j\|_{H^s}^2 - 2\langle\varphi_i,\varphi_j\rangle = 2\|\varphi\|_{H^s}^2 - 2\langle\varphi_i,\varphi_j\rangle.$$

The inner product

$$\begin{aligned}\langle \varphi_i, \varphi_j \rangle &= \int (1+|\xi|^2)^s \varphi_i(\xi) \overline{\varphi_j(\xi)} \, \mathrm{d}\xi \\ &= \int e^{-2\pi i \xi (a_i - a_j)} (1+|\xi|^2)^s |\varphi(\xi)|^2 \, \mathrm{d}\xi \\ &= \left(|\varphi|^2 (1+|\xi|^2)^s \right)^\wedge (a_i - a_j)\end{aligned}$$

tends to 0 as $j \to \infty$, thus

$$\|\varphi_i - \varphi_0\|_{H^s}^2 = 2\|\varphi\|_{H^s}^2,$$

which implies $\varphi = 0$ since $\|\varphi_i - \varphi_0\|_{H^s}^2$ is arbitrarily small, a contradiction! □