Solutions to Homework 10

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Folland. *Real Analysis*

Exercise 7.1.2

(1)

Proof. As a union of open sets, N is obviously open. Therefore, the inner regularity of μ implies

$$\mu(N) = \sup\{\mu(K) \mid K \subset N, K \text{ compact}\}.$$

Note that K is covered by finitely many open null sets, thus $\mu(K) = 0$ for every compact $K \subset N$. Consequently, we obtain $\mu(N) = 0$. □

(2)

Proof. $\Longleftarrow$:

If $x \notin \operatorname{supp} \mu$, then $x \in N$. Since $\{x\}$ itself is compact, there is a $f \in C_c(X)$ such that

$$0 \le \varphi \le 1,\ \varphi(x) = 1, \operatorname{supp} \varphi \subset N \Longrightarrow \int \varphi \,\mathrm{d}\mu = \int_N \varphi \,\mathrm{d}\mu = 0.$$

$\Longrightarrow$:

Suppose $x \in \operatorname{supp} \mu$ and $f \in C_c(X)$ in possession of the properties required in the problem. Let $f(x) = c > 0$, then its continuity implies $U_0 = f^{-1}(\frac{c}{2}, \frac{3c}{2})$ is open. Since $x \in U_0$, we have $\mu(U_0) > 0$, thus

$$\int f \,\mathrm{d}\mu \ge \int_{U_0} f \,\mathrm{d}\mu \ge \frac{c}{2}\mu(U_0) \ge 0.$$

□

Exercise 7.2.7

Proof. As is known, μ_A is a Borel measure. For a compact K, we have

$$\mu_E(K) = \mu(E \cap K) \leq \mu(K) < +\infty.$$

Since μ is σ Finite, there is a partition

$$X = \bigsqcup_{j=1}^{\infty} E_j$$

such that $\mu(E_j) < +\infty$, and μ is regular. As a result, we have

$$\mu(E) = \inf\{\mu(U) \mid U \text{ open}, E \subset U\} = \sup\{\mu(K) \mid K \text{ compact}, K \subset E\}$$

for any Borel set E.

Therefore, μ_E is outer regular since

$$\begin{aligned}\mu_A(E) = \mu(E \cap A) &= \inf\{\mu(U) \mid U \text{ open}, E \cap A \subset U\} \\ &= \inf\{\mu(U \cap A) \mid U \text{ open}, E \cap A \subset U \cap A\} \\ &= \inf\{\mu_A(U) \mid U \text{ open}, E \subset U\}.\end{aligned}$$

The inner regularity is similar. □

Exercise 7.2.8

Proof. It is obvious that ν is a Borel measure, and $\nu \ll \mu$.

Fix a Borel set $E \subset X$ and $\varepsilon > 0$, the absolute continuity implies the existence of $\delta > 0$, such that

$$\mu(U \backslash E) < \delta \Longrightarrow \nu(U \backslash E) < \varepsilon.$$

According to the outer regularity of μ, there exists an open $U \subset X$ including E such that $\mu(U \backslash E) < \delta$, which results in the outer regularity of ν. The inner regularity for open sets is similar. □

Exercise 7.2.9

(1)

Proof. For an open set U, we have

$$\begin{aligned}\nu'(U) &= \sup\left\{\int f\varphi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \le f \le 1, \operatorname{supp} f \in U\right\}\\ &= \sup\left\{\int \psi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \le \psi \le \varphi, \operatorname{supp} \psi \in U\right\}\\ &= \sup\left\{\int_U \psi\,\mathrm{d}\mu \,\middle|\, \psi \in C_c(X), 0 \le \psi \le \varphi\right\}\\ &= \int_U \varphi\,\mathrm{d}\mu\\ &= \nu(U).\end{aligned}$$

□

(2)

Proof. Since φ is continuous, the sets

$$V_k = \varphi^{-1}\left(2^{k-1}, 2^{k+1}\right),\ k \in \mathbb{Z}.$$

are open. Moreover, φ is positive, thus

$$X = \bigcup_{k=-\infty}^{+\infty} V_k.$$

Let E be a Borel set and define $E_k = E \cap V_k$. Note that φ is bounded in V_k, thus ν is absolutely continuous in respect to μ. Fix $\varepsilon > 0$, there exists $\delta_k > 0$ such that

$$\mu(U_k \backslash E_k) < \delta_k \Longrightarrow \nu(U_k \backslash E_k) < \frac{\varepsilon}{3 \cdot 2^{-|k|}}.$$

The outer regularity of μ implies the existence of an open U_k including E_k such that $\mu(U_k \backslash E_k) < \delta_k$.

Consider the open set

$$U = \bigcup_{k=-\infty}^{+\infty} U_k \supset \bigcup_{k=-\infty}^{+\infty} E_k \supset E,$$

we obtain

$$\nu(U \backslash E) \leq \sum_{k=-\infty}^{+\infty} \nu(U_k \backslash E_k) \leq \sum_{k=-\infty}^{+\infty} \frac{\varepsilon}{3 \cdot 2^{-|k|}} = \varepsilon,$$

namely the outer regularity of ν. □

(3)

Proof. Utilize (1) and (2), we obtain

$$\nu(E) = \inf\{\nu(U) \mid E \subset U, U \text{ open}\} = \inf\{\nu'(U) \mid E \subset U, U \text{ open}\} = \nu'(E)$$

for every Borel set E. □