Solutions to Homework 16

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Folland. *Real Analysis*

Exercise 9.1.13

(1)

Proof. Let η be the standard mollifier and $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$. Since $F * \eta_\varepsilon \in C^\infty$, we have

$$0 = \partial_j F * \eta_\varepsilon = \partial_j(F * \eta_\varepsilon) \Longrightarrow F * \eta_\varepsilon = C_\varepsilon.$$

For $f \in C_c^\infty$, we have

$$\langle F, f\rangle = \lim_{\varepsilon\to 0}\langle F, f * \eta_\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle F * \eta_\varepsilon, f\rangle = \lim_{\varepsilon\to 0}\langle C_\varepsilon, f\rangle,$$

which implies C_ε converges to some constant C. Let ε tend to 0, then $F = C$. □

Exercise 9.1.15

(1)

Proof. Let

$$G^\varepsilon(x,t) = G(x,t)\chi_{(\varepsilon,+\infty)}.$$

Since G^ε tends to $G \in L^1_{loc}$ pointwise, we conclude that $G^\varepsilon \to G$ in D'.

For $\varphi \in C_c^\infty$,

$$\begin{aligned}
\langle(\partial_t - \Delta)G^\varepsilon, \varphi\rangle &= \langle(\partial_t - \Delta)G^\varepsilon, \varphi\rangle \\
&= -\iint G(x,t)\chi_{(\varepsilon,+\infty)}(\partial_t + \Delta)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_\varepsilon^{+\infty}\left(\int \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{t}}(\partial_t + \Delta)\varphi(x,t)\,\mathrm{d}x\right)\mathrm{d}t \\
&= -\int_\varepsilon^{+\infty}\left(\int (\partial_t - \Delta)\frac{e^{-\frac{|x|^2}{t}}}{(4\pi t)^{\frac{n}{2}}}\varphi(x,t)\,\mathrm{d}x\right)\mathrm{d}t + \int \frac{e^{-\frac{|x|^2}{\varepsilon}}}{(4\pi\varepsilon)^{\frac{n}{2}}}\varphi(x,\varepsilon)\,\mathrm{d}x \\
&= \int \frac{1}{(4\pi\varepsilon)^{\frac{n}{2}}}e^{-\frac{|x|^2}{\varepsilon}}\varphi(x,\varepsilon)\,\mathrm{d}x \\
&\to \int \delta(x)\varphi(x,0)\,\mathrm{d}x \\
&= \varphi(0,0),
\end{aligned}$$

as $\varepsilon \to 0$, which implies

$$(\partial_t - \Delta)G = \delta.$$

□

(2)

Proof. By dominated convergence theorem,

$$\begin{aligned}(\partial_t - \Delta)f &= \partial_t \iint f(y,s)\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \Delta_x \iint f(y,s)\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \iint f(y,s)\partial_t\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \iint f(y,s)\Delta_x\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= -\iint f(y,s)\partial_s\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \iint f(y,s)\Delta_y\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \iint \partial_s f(y,s)\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad - \iint \Delta_y f(y,s)\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \iint \delta(x,t)\varphi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \varphi(x,t).\end{aligned}$$

□

Exercise 9.2.19

(1)

Proof. For $\varphi \in \mathcal{S}$,

$$\begin{aligned}\langle (F - F_0), \varphi\rangle &= \lim_{\varepsilon\to 0} \mathrm{P.V.} \int \left(\frac{1}{x} - \frac{x}{x^2+\varepsilon^2}\right)\varphi(x)\,\mathrm{d}x \\ &= \lim_{\varepsilon\to 0} \varepsilon^2 \mathrm{P.V.} \int \frac{\varphi(x)}{x(x^2+\varepsilon^2)}\,\mathrm{d}x \\ &\le \lim_{\varepsilon\to 0} \varepsilon^2 \mathrm{P.V.} \int \frac{\varphi(x)}{x^3}\,\mathrm{d}x \\ &= 0,\end{aligned}$$

since one can similarly verify by truncation that the principal value integral is bounded by semi-norms of φ. □

(2)

Proof. In the sense of weak * topology, (1) implies

$$\lim_{\varepsilon\to 0} G_\varepsilon = \lim_{\varepsilon\to 0} \frac{x \mp i\varepsilon}{x^2+\varepsilon^2} = F_0 \mp i \lim_{\varepsilon\to 0} \varepsilon \frac{1}{x^2+\varepsilon^2}.$$

We only need to compute the distributional limit of $\frac{\varepsilon}{x^2+\varepsilon^2}$.

In fact, dominated convergence theorem implies

$$\begin{aligned}
\lim_{\varepsilon\to 0}\langle \varepsilon(x^2+\varepsilon^2)^{-1}, \varphi\rangle &= \lim_{\varepsilon\to 0}\int \frac{\varepsilon\varphi(x)}{x^2+\varepsilon^2}\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int \frac{\varphi(\varepsilon y)}{y^2+1}\,\mathrm{d}y \\
&= \int \frac{\varphi(0)}{y^2+1}\,\mathrm{d}y \\
&= \pi\varphi(0).
\end{aligned}$$

□

(3)

Proof. By definition,

$$\begin{aligned}
\hat{S}_\varepsilon(\xi) &= \int e^{-2\pi\varepsilon|x|}\operatorname{sgn}(x) e^{-2\pi i x\xi}\,\mathrm{d}x \\
&= \int_0^{+\infty} e^{-2\pi x(\varepsilon+i\xi)}\,\mathrm{d}x - \int_{-\infty}^0 e^{2\pi x(\varepsilon-i\xi)}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\frac{1}{\varepsilon+i\xi} - \frac{1}{\varepsilon-i\xi}\right) \\
&= -\frac{i\xi}{\pi(\xi^2+\varepsilon^2)} \\
&= (\pi i)^{-1}F_\varepsilon(\xi),
\end{aligned}$$

which implies

$$(\operatorname{sgn})^\wedge = (\pi i)^{-1}F_0$$

in the sense of distribution. □

(4)

Proof. We have proved this conclusion in a previous exercise regarding Hilbert transform. □

(5)

Proof. For

$$\chi = \chi_{(0,+\infty)} = \frac{1}{2} + \frac{1}{2}\mathrm{sgn},$$

we have

$$\hat{\chi} = \frac{1}{2}\delta + \frac{1}{2}(\mathrm{sgn})^\wedge = \frac{1}{2}\delta + \frac{1}{2\pi i}F_0.$$

For

$$\chi = \chi_{(0,+\infty)} = \lim_{\varepsilon\to 0} e^{-\varepsilon x}\chi,$$

we have

$$\begin{aligned}
\hat{\chi}(x) &= \int \lim_{\varepsilon\to 0} e^{-\varepsilon x}\chi e^{-2\pi i x\xi}\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\int_0^{+\infty} e^{-x(\varepsilon-2\pi i\xi)}\,\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}(\varepsilon - 2\pi i\xi)^{-1} \\
&= \frac{1}{2\pi i}F_0(\xi) + \frac{1}{2}\delta(\xi).
\end{aligned}$$

□