Solutions to Homework 12

Yu Junao

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Folland. *Real Analysis*

Exercise 8.1.1

Proof. We only prove the first identity as similar approaches are applicable for the second one.

For $|\alpha| = 1$, without loss of generality assume $\alpha = (1, 0, \cdots, 0)$, then

$$\partial^\alpha \left(x^\beta f\right) = x^\beta \partial^\alpha f + \beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2} \cdots x_n^{\beta_n} f = x^\beta \partial^\alpha f + \sum_{\gamma, \delta} c_{\gamma\delta} x^\delta \partial^\gamma f$$

where

$$c_{\gamma\delta} = \begin{cases} \beta_1, & \beta_1 \geq 0, \gamma = 0, \delta = (\beta_1 - 1, \beta_2, \cdots, \beta_n), \\ 0, & \text{else.} \end{cases}$$

Assume the conclusion is correct for $|\alpha| = k - 1$. Assume $|\alpha| = k$ and $\alpha_1 \geq 1$, then for $\alpha' = (\alpha_1 - 1, \alpha_2, \cdots, \alpha_n)$ we have

$$\begin{aligned} \partial^\alpha \left(x^\beta f\right) = \partial_1 \partial^{\alpha'} \left(x^\beta f\right) &= \partial_1 \left(x^\beta \partial^\alpha f + \sum_{\gamma, \delta} c'_{\gamma\delta} x^\delta \partial^\gamma f \right) \\ &= x^\beta \partial^\alpha f + \beta_1 x^{\beta'} f + \sum_{\gamma, \delta} c'_{\gamma\delta} x^\delta \partial^\gamma f \\ &= x^\beta \partial^\alpha f + \sum_{\gamma, \delta} c_{\gamma\delta} x^\delta \partial^{(\gamma_1 + 1, \gamma_2, \cdots, \gamma_n)} f. \end{aligned}$$

By the assumption of induction, the conclusion is correct for every k. □

Exercise 8.1.2

(1)

Proof. We shall prove by induction on n.

As is known, the conclusion is correct for $n = 1, 2$. Assume it is correct for $n-1$, then for $x' = (x_1, \cdots, x_{n-1})$ we have

$$\begin{aligned}(x_1 + \cdots + x_n)^k &= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!}(x_1 + \cdots + x_{n-1})^j x_n^{k-j} \\ &= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x_n^{k-j} \left(\sum_{|\beta|=j} \frac{j!}{\beta!} x'^{\beta} \right) \\ &= \sum_{j=0}^{k} \sum_{|\beta|=j} \frac{k!}{(k-j)!\beta!} x'^{\beta} x_n^{k-j} \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} x'^{\beta} x_n^{k-j}\end{aligned}$$

since $|\alpha| = |\beta| + k - j = k$.

By the assumption of induction, the conclusion is correct for every n. ☐

(2)

Proof. Direct computation implies

$$\begin{aligned}(x+y)^{\alpha} &= \prod_{i=1}^{n} (x_i + y_i)^{\alpha_i} \\ &= \prod_{i=1}^{n} \sum_{j=1}^{\alpha_i} \frac{\alpha_i!}{j!(\alpha_i - j)!} x_i^j y_i^{\alpha_i - j} \\ &= \sum_{\beta+\gamma=\alpha} \prod_{i=1}^{n} \frac{\alpha_i!}{\beta_i!\gamma_i!} x_i^{\beta_i} y_i^{\gamma_i} \\ &= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^{\beta} y^{\gamma}\end{aligned}$$

☐

Exercise 8.3.14

Proof. It is easy to check that the inequality is invariant under translation and scaling, thus we assume $a = 0$ and $b = \frac{1}{2}$.

Set

$$F(x) = \begin{cases} f(x), & 0 \le x \le \frac{1}{2}, \\ -f(-x), & -\frac{1}{2} \le x < 0. \end{cases}$$

Note $F(x) \in C^1[-\frac{1}{2}, \frac{1}{2}]$ since

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{F(h)}{h} = f'(0),$$

and we only need to prove

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F(x)|^2 \, \mathrm{d}x \le \frac{1}{4\pi^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |F'(x)| \, \mathrm{d}x.$$

By Parseval's identity, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F(x)|^2 \, \mathrm{d}x = \sum_{n=-\infty}^{+\infty} |\hat{F}(n)|^2$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |F'(x)|^2 \, \mathrm{d}x = \sum_{n=-\infty}^{+\infty} \left| 2n\pi i \hat{F}(n) \right|^2 = 4\pi^2 \sum_{n=-\infty}^{+\infty} n^2 |\hat{F}(n)|^2.$$

Particularly, $\hat{F}(0) = 0$ since F is odd, thus the inequality holds, which achieves equality if and only if F satisfies $\hat{F}(n) = 0$ unless $n = \pm 1$. □

Exercise 8.4.28

(1)

Proof. Consider the partial sum of Poisson kernel. Note that

$$\left| f(y) \sum_{n=-N}^{N} r^{|n|} e^{2\pi i n(x-y)} \right| \le |f(x)| \sum_{n=-N}^{N} r^{|n|} \le 2|f(x)| \sum_{n=1}^{\infty} r^n = \frac{2r}{1-r} |f(x)| < +\infty$$

for $0 < r < 1$.

Therefore, dominated convergence theorem implies

$$\begin{aligned}(f * P_r)(x) &= \int f(y) \sum_{n=-\infty}^{+\infty} r^{|n|} e^{2n\pi i(x-y)} \,\mathrm{d}y \\ &= \int \lim_{N\to\infty} f(y) \sum_{n=-N}^{N} r^{|n|} e^{2n\pi i(x-y)} \,\mathrm{d}y \\ &= \lim_{N\to\infty} \int \sum_{n=-N}^{N} f(y) r^{|n|} e^{2n\pi i(x-y)} \,\mathrm{d}y \\ &= A_r f(x).\end{aligned}$$

□

(2)

Proof. By direct computation,

$$\begin{aligned}P_r(x) &= 1 + \sum_{n=1}^{\infty} r^n \left(e^{2n\pi ix} + e^{-2n\pi ix}\right) \\ &= 1 + \sum_{n=1}^{\infty} \left(re^{2\pi ix}\right)^n + \sum_{n=1}^{\infty} \left(re^{-2\pi ix}\right)^n \\ &= 1 + \frac{re^{2\pi ix}}{1 - re^{2\pi ix}} + \frac{re^{-2\pi ix}}{1 - re^{-2\pi ix}} \\ &= \frac{1 - r^2}{1 + r^2 - 2r\cos 2\pi x}.\end{aligned}$$

□

Exercise 8.4.32

Proof. $\Longrightarrow$:

Define

$$g(z) = g(e^{2\pi ix}) = f(x)$$

for $x \in \mathbb{T}$, then $g(z)$ is an analytic function on $\mathbb{S}^1$. By the analyticity of exponential function, g is naturally extended to a holomorphic function on an annulus

$$A = \{z \mid 1 - \delta < |z| < 1 + \delta\}.$$

Consider the Laurent expansion at 0

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n z^n,$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{g(z)}{z^{n+1}} \,\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_0^1 g(e^{2\pi i x}) e^{-2n\pi i x} \,\mathrm{d}x \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} f(x) e^{-2n\pi i x} \,\mathrm{d}x \\ &= \frac{1}{2\pi i} \hat{f}(n) \end{aligned}$$

as a result of residue theorem.

Take $z_0 \in A$ such that $|z_0| > 1$, then the convergence of Laurent series in A implies

$$\lim_{n\to\infty} \left| \frac{1}{2\pi i} \hat{f}(n) z_0^n \right| = 0 \Longrightarrow \lim_{n\to\infty} |\hat{f}(n)||z_0|^n = 0,$$

hence there exists $N_1 > 0$ and $C_1 > 0$ such that

$$|\hat{f}(n)| \leq C_1 e^{-\varepsilon_1 n}, \ n \geq N_1.$$

where $\varepsilon_1 = \log |z_0| > 0$.

Take $z_0' \in A$ such that $|z_0'| < 1$, and the same argument implies the existence of $N_2 > 0$ and $C_2 > 0$ such that

$$|\hat{f}(n)| \leq C_2 e^{-\varepsilon_2 n}, \ n \leq -N_2.$$

where $\varepsilon_2 = -\log |z_0'| > 0$.

Let

$$\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$$

and

$$C = \max\left\{ C_1, C_2, \max\left\{ \frac{\hat{f}(1-N_2)}{e^{1-N_2}}, \cdots, \frac{\hat{f}(-1)}{e^{-1}}, \hat{f}(0), \frac{\hat{f}(1)}{e}, \cdots, \frac{\hat{f}(N_1-1)}{e^{N_1-1}} \right\} \right\}. \tag{1}$$

then we have

$$|\hat{f}(n)| \leq Ce^{-n}$$

for $n \in \mathbb{Z}$.

$\Longleftarrow$:

Consider

$$\begin{aligned} A_r f(x) &= \hat{f}(0) + \sum_{n=1}^{\infty} r^n \left(\hat{f}(n) e^{2n\pi ix} + \hat{f}(-n) e^{-2n\pi ix} \right) \\ &= \hat{f}(0) + \sum_{n=1}^{\infty} r^n \left(\hat{f}(n) z^n + \hat{f}(-n) z^{-n} \right). \end{aligned}$$

The inequality yields an uniform boundedness estimate

$$\begin{aligned} |A_r f(x)| &\leq |\hat{f}(0)| + C \sum_{n=1}^{\infty} r^n e^{-\varepsilon n} \left| e^{2n\pi ix} + e^{2n\pi ix} \right| \\ &\leq |\hat{f}(0)| + 2C \sum_{n=1}^{\infty} (re^{-\varepsilon})^n \\ &\leq |\hat{f}(0)| + \frac{2Cr}{e^{\varepsilon} - r} \\ &\leq |\hat{f}(0)| + \frac{2C}{e^{\varepsilon} - 1} \\ &< +\infty \end{aligned}$$

for $0 < r < \frac{e^{\varepsilon}+1}{2}$, then this Laurent series converges, namely $A_r f(x)$ corresponds to a holomorphic function on $\mathbb{S}^2$ for every $r \in (0, \frac{e^{\varepsilon}+1}{2})$ including $r = 1$.

Since $A_1 f(x) = f(x)$, the complex-variable function $g(z)$ define at the start is holomorphic on $\mathbb{S}^1$, resulting in the analyticity of $f(x)$ on $\mathbb{T}$. $\square$