Solutions to Homework 06

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Folland. ***Real Analysis***

Exercise 6.1.7

Proof. By interpolation inequality,

$$\|f\|_q \leq \|f\|_p^{\frac{p}{q}} \|f\|_\infty^{\frac{q-p}{q}} < +\infty \Longrightarrow f \in L^q.$$

As an immediate corollary, we have

$$\overline{\lim_{q\to\infty}} \|f\|_q \leq \|f\|_\infty.$$

On the other hand, consider the level set

$$A_\lambda = \{|f| \geq \lambda\}$$

for $0 < \lambda < \|f\|_\infty$. Therefore,

$$\lambda^q \mu(E_\lambda) = \int_{E_\lambda} \lambda^q \,\mathrm{d}\mu \leq \int_{E_\lambda} |f|^q \,\mathrm{d}\mu \leq \|f\|_q^q.$$

For fixed λ,

$$\|f\|_q \geq \lambda \left(\mu(E_\lambda)\right)^{\frac{1}{q}} \Longrightarrow \underline{\lim_{q\to\infty}} \|f\|_q \geq \lambda.$$

Let λ tend to $\|f\|_\infty$, and

$$\underline{\lim_{q\to\infty}} \|f\|_q \geq \|f\|_\infty \geq \overline{\lim_{q\to\infty}} \|f\|_q.$$

As a result,

$$\lim_{q\to\infty} \|f\|_q = \|f\|_\infty.$$

□

Exercise 6.1.9

Proof. Fix a $\varepsilon > 0$, we have

$$\begin{aligned}\mu\{|f_n - f| \geq \varepsilon\} &= \frac{1}{\varepsilon^p}\int_{\{|f_n-f|^p\geq\varepsilon^p\}} \varepsilon^p \,\mathrm{d}\mu \\ &= \frac{1}{\varepsilon^p}\int_{\{|f_n-f|^p\geq\varepsilon^p\}} |f_n - f|^p \,\mathrm{d}\mu \\ &\leq \frac{1}{\varepsilon^p}\int_X |f_n - f|^p \,\mathrm{d}\mu \\ &\to 0,\ n\to\infty.\end{aligned}$$

Conversely, assume $\{f_n\}_{n=1}^{\infty}$ does not converge to f in L^p, then there is a subsequence $\{g_n\}_{n=1}^{\infty} \subset \{f_n\}_{n=1}^{\infty}$ such that $\exists\, \varepsilon_0 > 0$,

$$\|g_n - f\|_p \geq \varepsilon,\ \forall\, n.$$

Since $g_n \to f$ in measure, there is a subsequence $\{h_n\}_{h=1}^{\infty} \subset \{g_n\}_{n=1}^{\infty}$ that converges to f almost everywhere. Dominant convergence theorem implies $h_n \to f$ in L^p, a contradiction. □

Exercise 6.1.10

Proof. $\Longrightarrow$:

The triangle inequality implies

$$|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p \to 0,\ n\to\infty.$$

$\Longleftarrow$:

As for the inverse proposition, we shall verify a primary inequality first

$$|a \pm b|^p \leq 2^{p-1}\left(|a|^p + |b|^p\right),\ \forall, p \geq 1.$$

In fact, it is equivalent with

$$\left|\frac{a\pm b}{2}\right|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p,$$

a consequence of the convexity of function $f(x) = |x|^p$ for $p \geq 1$.

Back to the point, construct

$$g_n = 2^{p-1}\left(|f_n|^p + |f|^p\right)$$

that converges to $g = |2f|^p \in L^1$ almost everywhere. Since $|f_n - f|^p \leq g_n$, the dominant convergence theorem implies

$$\lim_{n\to\infty}\int |f_n - f|^p = \int \lim_{n\to\infty} |f_n - f|^p = 0 \Longrightarrow \lim_{n\to\infty} \|f_n\|_p = \|f\|_p.$$

□

Exercise 6.1.15

Proof. $\Longrightarrow$:

The completeness of L^p space implies $\{f_n\}_{n=1}^{\infty}$ converges to some $f \in L^p$, and the first two conclusions are immediate due to our previous homework.

As for the third, consider an increasing sequence of sets

$$E_m = \left\{ |f_n| \geq \frac{1}{m} \right\}$$

for fixed n. Obviously, $\mu(E_m)$ is finite since $f_n \in L^p$. Additionally, note that

$$\int_E |f_n|^p = \int |f_n|^p < +\infty,$$

where

$$E = \bigcup_{m=1}^{\infty} E_m = \{|f_n| \geq 0\}.$$

Due to the fact that $|f_n \chi_{E_m^c}| \leq f_n \in L^p$, we can apply the dominant convergence theorem,

$$\lim_{m\to\infty} \|f_n\|_{L^p(E_m^c)} = \lim_{m\to\infty} \left(\int_{E_m^c} |f_n|^p \right)^{\frac{1}{p}} = \left(\lim_{m\to\infty} \int |f_n|^p \chi_{E_m^c} \right)^{\frac{1}{p}} = \left(\int_{E^c} |f_n| \right)^{\frac{1}{p}} = 0.$$

As a result, $\forall\, \varepsilon > 0$, $\exists\, m > 0$, such that

$$\|f_n\|_{L^p(E_m^c)}^p < \varepsilon \Longrightarrow \int_{E_m^c} |f_n|^p < \varepsilon,$$

while $\mu(E_m^c) < +\infty$.

$\Longleftarrow$:

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of L^p functions in possession of the three properties. For a fixed $\forall\, \varepsilon > 0$, consider the sets

$$A_{mn} = E \cap \left\{ |f_m - f_n| \geq \frac{\varepsilon^{\frac{1}{p}}}{3^{\frac{1}{p}} \mu(E)} \right\}.$$

Here E is of finite measure and

$$\int_{E^c} |f_n|^p < \left(\frac{\varepsilon}{3} \right)^p, \ \forall\, n.$$

Provided with such conditions, we have

$$\int_{E\backslash A_{mn}} |f_m - f_n|^p \leq \int_{E\backslash A_{mn}} \frac{\varepsilon}{3\mu(E)} = \frac{\varepsilon\mu(E\backslash A_{mn})}{3\mu(E)} \leq \frac{\varepsilon}{3}.$$

Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy in measure, we assume $\mu(A_{mn}) < \delta$ is small in measure for sufficiently large m, n. As a result,

$$\int_{A_{mn}} |f_m - f_n|^n \leq 2^{p-1} \int_{A_{mn}} |f_m|^p + 2^{p-1} \int_{A_{mn}} |f_n|^p \leq \frac{\varepsilon}{3}.$$

The last inequality is correct for sufficiently small δ as a consequence of the uniform integrability of $\{f_n\}_{n=1}^{\infty}$.

Combine the inequalities above, we ultimately obtain

$$\|f_m - f_n\|_p \leq \left(\int_{E^c} |f_m - f_n|^p + \int_{E\backslash A_{mn}} |f_m - f_n|^p + \int_{A_{mn}} |f_m - f_n|^p \right) \leq \varepsilon.$$

□