Solutions to Homework 08

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Folland. *Real Analysis*

Exercise 6.2.20

(1)

Proof. Denote

$$M = \max \left\{ \sup_n \|f_n\|_p, \|f\|_p \right\} < +\infty.$$

Fix an arbitrary $g \in L^{p'}$ and an $\varepsilon > 0$. As a result of the absolute continuity of Lebesgue integral, there exists $\delta > 0$, such that

$$\mu(E) < \delta \Longrightarrow \left(\int_E |g|^{p'} \right)^{\frac{1}{p'}} < \frac{\varepsilon}{4M}.$$

Similarly, there is a set A of finite measure such that

$$\left(\int_{A^c} |g|^{p'} \right) < \frac{\varepsilon}{4M}.$$

Moreover, Egorov's theorem implies the existence of a subset $B \subset A$ such that $f_n \rightrightarrows f$ on B.

With the preparations above, we shall prove the weak convergence of $\{f_n\}_{n=1}^{\infty}$. Hölder's inequality implies

$$\begin{aligned}\int_{B^c} |(f_n - f)g| &\leq \left(\int_{B^c} |f_n - f|^p \right)^{\frac{1}{p}} \left(\int_{B^c} |g|^{p'} \right)^{\frac{1}{p'}} \\ &\leq (\|f_n\|_p + \|f\|_p) \left(\left(\int_{A \setminus B} |g|^{p'} \right)^{\frac{1}{p'}} + \left(\int_{A^c} |g|^{p'} \right)^{\frac{1}{p'}} \right) < \frac{\varepsilon}{2}.\end{aligned}$$

On the other hand, we further have

$$\int_B |(f_n - f)g| \le \|f_n - f\|_p \|g\|_{p'} \le \frac{\varepsilon}{2}.$$

Here n is so large such that

$$|f_n - f| \le \frac{\varepsilon}{2M\mu(B)}.$$

To sum up, we obtain

$$\int |(f_n - f)g| < \varepsilon \to 0$$

as n tends to infinity. □

(2)

Proof. Consider $f_n = \chi_{[n,n+1]}$, which converges to $f = 0$ almost everywhere. The function $g = 1$ belongs to L^∞, but

$$\int f_n g = \int f_n = 1 \ne 0.$$

Now assume $\{f_n\}_{n=1}^\infty$ is bounded in L^∞ and $g \in L^1$ where the measure μ is σ-finite. Note that

$$\int (f_n - f)g \le (\|f_n\| + \|f\|) \int |g| < +\infty,$$

by dominant convergence theorem we have

$$\lim_{n\to\infty} \int (f_n - f)g = \int \lim_{n\to\infty} (f_n - f)g = 0.$$

□

Exercise 6.2.22

(1)

Proof. As Riemann-Lebesgue lemma implies, if $f \in L^2(X)$, then

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(2\pi n x)\,\mathrm{d}x = 0.$$

Therefore, $\{\cos(2\pi nx)\}_{n=1}^{\infty}$ converges weakly to 0.

For $\varepsilon \in (0,1)$, it is easy to check that

$$\mu\left\{|\cos(2\pi nx)| > \frac{1}{2}\right\} = \frac{2}{3},$$

independent of n.

Suppose that $\{\cos(2\pi nx)\}_{n=1}^{\infty}$ converges to 0 almost everywhere. By dominant convergence theorem, we have

$$0 = \int_0^1 \lim_{n\to\infty} \cos^2(2\pi nx)\,\mathrm{d}x = \frac{1}{2}\lim_{n\to\infty}\int_0^1 (1+\cos(4\pi nx))\,\mathrm{d}x = \frac{1}{2},$$

a contradiction! □

(2)

Proof. Since f_n converges to 0 pointwise except for $x = 0$, we conclude that $f_n \to 0$ almost everywhere. For $\varepsilon > 0$, we have

$$\mu\{|f_n| > \varepsilon\} \le \frac{1}{n} \to 0,\ n \to \infty,$$

thus $f_n \to 0$ in measure.

It is obvious that $g = \chi_{[0,1]}$ belongs to L^p for all $p \in [1, +\infty]$, but

$$\int f_n g = \int f_n = 1 > 0.$$

Therefore, f_n never converges weakly in L^p. □

Exercise 6.5.41

Proof. Without loss of generality suppose $p < q$. T is a linear operator since

$$\begin{aligned}\int T(\lambda_1 f_1 + \lambda_2 f_2)g &= \int (\lambda_1 f_1 + \lambda_2 f_2)Tg \\ &= \lambda_1 \int f_1 Tg + \lambda_2 \int f_2 Tg \\ &= \int (\lambda_1 Tf_1 + \lambda_2 Tf_2)g,\ \forall\, g \in L^p \cap L^q.\end{aligned}$$

We first show that T is bounded on L^q. By the duality expression of L^q norm, we have for $f \in L^q$ that

$$\begin{aligned}
\|Tf\|_q &= \sup\left\{ \int (Tf)g \,\middle|\, g \in L^p, \|g\|_p = 1 \right\} \\
&= \sup\left\{ \int f(Tg) \,\middle|\, g \in L^p, \|g\|_p = 1 \right\} \\
&\leq \sup\left\{ \|f\|_q \|Tg\|_p \mid g \in L^p, \|g\|_p = 1 \right\} \\
&\leq \sup\left\{ \|T\|_{p\to p} \|f\|_q \|g\|_p \mid g \in L^p, \|g\|_p = 1 \right\} \\
&= \|T\|_{p\to p} \|f\|_q.
\end{aligned}$$

Riesz-Thorin interpolation theorem implies T is also bounded on L^r.

It is easy to verify that L^p is dense in L^r. Additionally, T is continuous since it is linear and bounded. As a result, the extension is unique. □

Exercise 6.5.42

Proof. Provided

$$\begin{aligned}
\lambda_{Tf}(\alpha) &\leq \frac{C}{\alpha^{q_0}} \|f\|_p^{q_0} \\
\lambda_{Tf}(\alpha) &\leq \frac{C}{\alpha^{q_1}} \|f\|_p^{q_1}
\end{aligned}$$

and

$$\frac{1}{q} = \frac{t}{q_0} + \frac{1-t}{q_1},$$

we have

$$\begin{aligned}
\|Tf\|_q^q &= q \int_0^{+\infty} \alpha^{q-1} \lambda_{Tf}(\alpha) \,\mathrm{d}\alpha \\
&= q \int_0^{\|f\|_p} \alpha^{q-1} \lambda_{Tf}(\alpha) \,\mathrm{d}\alpha + q \int_{\|f\|_p}^{+\infty} \alpha^{q-1} \lambda_{Tf}(\alpha) \,\mathrm{d}\alpha \\
&\leq Cq \int_0^{\|f\|_p} \alpha^{q-q_0-1} \|f\|_p^{q_0} \,\mathrm{d}\alpha + Cq \int_{\|f\|_p}^{+\infty} \alpha^{q-q_1-1} \|f\|_p^{q_0} \,\mathrm{d}\alpha \\
&= C\|f\|_p^{q_0} \int_0^{\|f\|_p} \alpha^{q-q_0-1} \,\mathrm{d}\alpha + C\|f\|_p^{q_0} \int_{\|f\|_p}^{\infty} \alpha^{q-q_1-1} \,\mathrm{d}\alpha \\
&= C\|f\|_p^q.
\end{aligned}$$

□