Solutions to Homework 05

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Folland. *Real Analysis*

Exercise 3.1.3

(1)

Proof. If $f \in L^1(|\nu|)$, then

$$\int |f| \, \mathrm{d}\nu^+ \leq \int |f| \, \mathrm{d}\nu^+ + \int |f| \, \mathrm{d}\nu^- = \int |f| \, \mathrm{d}|\nu| < +\infty,$$
$$\int |f| \, \mathrm{d}\nu^- \leq \int |f| \, \mathrm{d}\nu^+ + \int |f| \, \mathrm{d}\nu^- = \int |f| \, \mathrm{d}|\nu| < +\infty.$$

If $f \in L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$, then

$$\int |f| \, \mathrm{d}|\nu| = \int |f| \, \mathrm{d}\nu^+ + \int |f| \, \mathrm{d}\nu^- < +\infty.$$

□

(2)

Proof.

$$\begin{aligned} \left| \int f \, \mathrm{d}\nu \right| &= \left| \int f \, \mathrm{d}\nu^+ - \int f \, \mathrm{d}\nu^- \right| \\ &\leq \left| \int f \, \mathrm{d}\nu^+ \right| + \left| \int f \, \mathrm{d}\nu^- \right| \\ &\leq \int |f| \, \mathrm{d}\nu^+ + \int |f| \, \mathrm{d}\nu^+ \\ &= \int |f| \, \mathrm{d}|\nu|. \end{aligned}$$

□

(3)

Proof. On the one hand,

$$\sup\left\{\left|\int_E f\,\mathrm{d}\nu\right| : |f|\le 1\right\} \le \sup\left\{\left|\int_E |f|\,\mathrm{d}|\nu|\right| : |f|\le 1\right\} \le |\nu|(E).$$

On the other hand, the supermum is achieved by

$$f(x) = \begin{cases} 1, & x\in E\cap P, \\ -1, & x\in E\cap N, \end{cases}$$

where P and N are respectively the positive and negative set of ν. □

Exercise 3.2.9

Proof. If $\nu_j \perp \mu$ for any j, then

$$\mu(E)\ne 0 \Longrightarrow \nu_j(E)=0,\ \forall\, j \Longrightarrow \sum_{j=1}^{\infty}\nu_j(E)=0,$$
$$\nu_j(E)>0,\ \forall\, j \Longrightarrow \mu(E)=0.$$

Therefore, μ is singular with respect to the summation of ν_j.

If $\nu_j \ll \mu$ for any j, then

$$\mu(E)=0 \Longrightarrow \nu_j(E)=0,\ \forall\, j \Longrightarrow \sum_{j=1}^{\infty}\nu_j(E)=0.$$

Therefore, the summation of ν_j is absolutely continuous with respect to μ. □

Exercise 3.2.11

(1)

Proof. Let $\{f_k\}_{j=1}^{n}$ be a finite subset of $L^1(\mu)$. By the definition, $\forall\,\varepsilon>0$, $\exists\,\delta_j>0$ for $1\le j\le n$, such that

$$\left|\int_E f_j\,\mathrm{d}\mu\right| < \varepsilon$$

as $\mu(E)<\delta_j$. Therefore, we can choose $\delta=\min\{\delta_1,\cdots,\delta_n\}$, which satisfies

$$\left|\int_E f_j\,\mathrm{d}\mu\right| < \varepsilon,\ \forall\, 1\le j\le n.$$

□

(2)

Proof. By definition, $\forall\, \varepsilon > 0$, $\exists\, N > 0$, such that

$$\left|\int (f_n - f)\,\mathrm{d}\mu\right| < \frac{\varepsilon}{2},\ \forall\, n > N.$$

As is proved above, $\exists\, \delta > 0$ such that

$$\left|\int_E f\,\mathrm{d}\mu\right| < \frac{\varepsilon}{2}, \quad \left|\int_E f_j\,\mathrm{d}\mu\right| < \varepsilon,\ \forall\, 1 \le j \le N$$

as $\mu(E) < \delta$.

To sum up, we have for $j \le N$ that

$$\left|\int_E f_j\,\mathrm{d}\mu\right| < \delta,$$

and for $j > N$ that

$$\left|\int_E f_j\,\mathrm{d}\mu\right| \le \left|\int_E f_j - f\,\mathrm{d}\mu\right| + \left|\int_E f\,\mathrm{d}\mu\right| \le \left|\int f_j - f\,\mathrm{d}\mu\right| + \left|\int_E f\,\mathrm{d}\mu\right| < \varepsilon.$$

Note that δ is independent of j, which implies that $\{f_j\}_{j=1}^{\infty}$ is uniformly integrable. □

Exercise 3.2.17

Proof. Consider a new measure ρ such that $\mathrm{d}\rho = f\,\mathrm{d}\mu$. For $E \in \mathcal{N}$, we have

$$\nu(E) = 0 \Longrightarrow \mu(E) = 0 \Longrightarrow \rho(E) = 0.$$

which implies $\rho \ll \nu$.

Let $g = \frac{\mathrm{d}\rho}{\mathrm{d}\nu}$ be the Radon-Nikodym derivative which is unique, and it is easy to check that

$$\int_E f\,\mathrm{d}\mu = \int_E g\,\mathrm{d}\nu.$$

□