Solutions to Homework 02

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Folland. *Real Analysis*

Exercise 1.2.1

(1)

Proof. Let $\mathcal{R}$ be a ring including $E_1, E_2, \cdots, E_n$, then

$$E_1 \cap E_2 = E_1 \backslash (E_1 \backslash E_2) \in \mathcal{R}.$$

Through induction, it is easy to check the intersection of this n sets belongs to $\mathcal{R}$.

If $\mathcal{R}$ is a σ-ring including $\{E_k\}_{k=1}^{\infty}$, we consider

$$E = \bigcup_{k=1}^{\infty} E_k \in \mathcal{R},$$

then

$$\begin{aligned} \bigcap_{k=1}^{\infty} E_k &= E \backslash \left(E \backslash \bigcap_{k=1}^{\infty} E_k \right) = E \backslash \left(E \cap \left(\bigcap_{k=1}^{\infty} E_k \right)^c \right) \\ &= E \backslash \left(E \cap \left(\bigcup_{k=1}^{\infty} E_k^c \right) \right) = E \backslash \left(\bigcup_{k=1}^{\infty} E \cap E_k^c \right) = E \backslash \left(\bigcup_{k=1}^{\infty} (E \backslash E_k) \right) \in \mathcal{R}. \end{aligned}$$

□

(2)

Proof. Let $\mathcal{R}$ be a ring. It is easy to check

$$\mathcal{R} \text{ is an algebra} \Longrightarrow \forall\, E \in \mathcal{R}, E^c \in \mathcal{R} \Longrightarrow X = E \cup E^c \in \mathcal{R},$$

and conversely,

$$X \in \mathcal{R} \Longrightarrow \forall\, E \in \mathcal{R}, E^c \in \mathcal{R} \Longrightarrow E^c = X \backslash E \in \mathcal{R} \Longrightarrow \mathcal{R} \text{ is a ring}.$$

This argument is still correct for σ-rings. □

(3)

Proof. Let
$$Y = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}.$$
It is obvious that Y is closed under complement since $(E^c)^c = E$.

For $\{E_k\}_{k=1}^{\infty} \subset Y$, we consider
$$A = \bigcup_{\substack{k \geq 1 \\ E_k \in \mathcal{R}}} E_k \in \mathcal{R}, \qquad B = \bigcap_{\substack{k \geq 1 \\ E_k^c \in \mathcal{R}}} E_k^c \in \mathcal{R},$$
then
$$\bigcup_{k=1}^{\infty} E_k = \left(\bigcup_{\substack{k \geq 1 \\ E_k \in \mathcal{R}}} E_k \right) \cup \left(\bigcup_{\substack{k \geq 1 \\ E_k^c \in \mathcal{R}}} E_k \right) = A \cup B^c = (A^c \cap B)^c = (B \backslash A)^c .$$

Therefore,
$$B \backslash A \in \mathcal{R} \Longrightarrow \bigcup_{k=1}^{\infty} E_k \in Y$$

□

(4)

Proof. Let
$$Z = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\},$$
and $E \in Z, F \in \mathcal{R}$ be arbitrary, then
$$E^c \cap F = F \backslash E = F \backslash (E \cap F) \in \mathcal{R} \Longrightarrow E^c \in Z.$$

Moreover, for $\{E_k\}_{k=1}^{\infty} \subset Y$ and arbitrary $F \in \mathcal{R}$, we have
$$\left(\bigcup_{k=1}^{\infty} E_k \right) \cap F = \bigcup_{k=1}^{\infty} (E_k \cap F) \in \mathcal{R} \Longrightarrow \bigcup_{k=1}^{\infty} E_k \in Z.$$

□

Exercise 1.3.6

Proof. First, we are going to show that $\overline{\mathcal{M}}$ is a σ-algebra. For $E \in \mathcal{M}, F \subset N \in \mathcal{N}$, we assume that $E \cap N = \varnothing$. Otherwise, we can substitute F, N respectively with $F \backslash E, N \backslash E$. Thus,

$$(E \cup F)^c = ((E \cup N) \backslash (N \backslash F))^c = (E \cup N)^c \cup (N^c \cup F) \in \overline{\mathcal{M}},$$

since $(E \cup N)^c \cup N^c \in \mathcal{M}$.

For such a sequence of sets $\{E_k\}_{k=1}^{\infty}, \{F_k\}_{k=1}^{\infty}, \{N_k\}_{k=1}^{\infty}$, we have

$$\left(\bigcup_{k=1}^{\infty} F_k \right) \subset \left(\bigcup_{k=1}^{\infty} N_k \right) \in \mathcal{N} \Longrightarrow = \left(\bigcup_{k=1}^{\infty} E_k \right) \cup \left(\bigcup_{k=1}^{\infty} F_k \right) \in \overline{\mathcal{M}}.$$

Next, we need to show $\bar{\mu}$ is a complete measure. Obviously,

$$\bar{\mu}(\varnothing) = \mu(\varnothing) = 0.$$

Additionally, for disjoint $\{E_k \cup F_k\}_{k=1}^{\infty}$,

$$\begin{aligned} \bar{\mu} \left(\bigcup_{k=1}^{\infty} (E_k \cup F_k) \right) &= \bar{\mu} \left(\left(\bigcup_{k=1}^{\infty} E_k \right) \cup \left(\bigcup_{k=1}^{\infty} F_k \right) \right) \\ &= \mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k \cup F_k). \end{aligned}$$

Therefore, $\bar{\mu}$ is a measure. Its definition implies completeness since for $F \subset N \in \mathcal{N}$, we have

$$\bar{\mu}(F) = \bar{\mu}(\varnothing \cup F) = \mu(\varnothing) = 0.$$

Finally, the uniqueness is left to be proved. Suppose there is another complete measure $\bar{\mu}'$ extending μ, we have

$$\left. \begin{aligned} \bar{\mu}'(E \cup F) \leq \bar{\mu}'(E \cup N) = \bar{\mu}'(E) \\ \bar{\mu}'(E \cup F) \geq \bar{\mu}'(E \cup \varnothing) = \bar{\mu}'(E) \end{aligned} \right\} \Longrightarrow \bar{\mu}' = \bar{\mu}, \ \forall\, E \in \mathcal{M}, F \subset N \in \mathcal{N}.$$

$\square$

Exercise 1.3.8

Proof. By definition,

$$\begin{aligned}\mu\left(\liminf_{j\to\infty} E_j\right) &= \mu\left(\bigcup_{k=1}^{\infty}\bigcap_{j=k}^{\infty} E_j\right) = \mu\left(\lim_{k\to\infty}\bigcap_{j=k}^{\infty} E_j\right) \\ &= \lim_{k\to\infty}\mu\left(\bigcap_{j=k}^{\infty} E_j\right) \le \lim_{k\to\infty}\inf_{j\ge k}\mu(E_j) = \liminf_{j\to\infty}\mu(E_j).\end{aligned}$$

We can similarly proof the other inequality. □

Exercise 1.3.10

Proof. Obviously,

$$\mu_E(\varnothing) = \mu(\varnothing\cap E) = \mu(\varnothing) = 0.$$

Let $\{A_k\}_{k=1}^{\infty} \subset \mathcal{M}$ be a sequence of disjoint sets, then $\{A_k \cap E\}_{k=1}^{\infty}$ are disjoint. Thus,

$$\begin{aligned}\mu_E\left(\bigsqcup_{k=1}^{\infty} A_k\right) &= \mu\left(\left(\bigsqcup_{k=1}^{\infty} A_k\right)\cap E\right) = \mu\left(\bigsqcup_{k=1}^{\infty}(A_k\cap E)\right) \\ &= \sum_{k=1}^{\infty}\mu(A_k\cap E) = \sum_{k=1}^{\infty}\mu_E(A_k).\end{aligned}$$

Therefore, μ_E is a measure. □