Solutions to Homework 09

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Folland. *Real Analysis*

Exercise 6.2.21

Proof. $\Longleftarrow$:

It has been proved in **exercise 6.2.20**.

$\Longrightarrow$:

Consider test function $\chi_{\{a\}} \in l^{p'}(A)$, the weak convergence of $\{f_n\}_{n=1}^{\infty}$ implies

$$f_n(a) = \int f_n \chi_{\{a\}} \to 0,\ n \to \infty.$$

Therefore, $f_n \to f$ pointwise.

According to the duality property of $l^p(A)$, we regard $\{f_n\}_{n=1}^{\infty}$ as a sequence of bounded linear functionals on $l^{p'}(A)$. For a fixed $g \in l^{p'}$, we have

$$\lim_{n\to\infty} \int f_n g \to 0 \Longrightarrow \sup_n \left| \int f_n g \right| < +\infty.$$

By resonance theorem, the operator norm, namely the l^p norm of $\{f_n\}$ is uniformly bounded. In other words, we obtain

$$\sup_n \|f\|_p < +\infty$$

□

Supplements

Exercise 1 (Urysohn's lemma). *Given an open $\Omega \subset \mathbb{R}^n$ and a compact set $K \subset \Omega$. Prove: there exists $\varphi \in C_c^{\infty}(\Omega)$ such that $\varphi|_K = 1$ and $0 \leq \varphi \leq 1$.*

Proof. Without loss of generality assume Ω is bounded. Otherwise, there exists $r > 0$ such that $K \subset B_r(0)$, thus we can substitute Ω with $\Omega \cap B_r(0)$ and let $\varphi = 0$ in $\Omega \backslash \overline{B_r(0)}$.

The boundedness of Ω admits the existence of a sufficiently large $R > 0$ such that $\Omega \subset B_R(0)$. It is easy to verify that $\overline{B_R(0)} \backslash \Omega$ is a bounded closed set, and then a compact set.

Now we are going to prove there is a positive distance between K and $\overline{B_R(0)} \backslash \Omega$. If

$$\inf \left\{ |x - y| \,\middle|\, x \in K, y \in \overline{B_R(0)} \backslash \Omega \right\} = 0,$$

then there is a sequence $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $|x_n - y_n| < \frac{1}{n}$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, it possesses a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, converging to $x \in K$. Moreover, there is a convergent subsequence of $\{y_{n_k}\}_{k=1}^{\infty}$, converging to $y \in \overline{B_R(0)} \backslash \Omega$. As a result, we have

$$x = y \Longrightarrow \overline{B_R(0)} \backslash \Omega \cap K \neq \varnothing \Longrightarrow \Omega^c \cap K \neq \varnothing \Longrightarrow \Omega^c \cap \Omega \neq \varnothing,$$

a contradiction.

Back to the point, take the mollifier η_ε such that $\varepsilon < \frac{d}{2}$, and define

$$\varphi(x) = \int \chi_{\tilde{K}}(y) \eta_\varepsilon(x - y) \, \mathrm{d}y \in C_c^\infty(\Omega).$$

where $K \subset \tilde{K} \subset \Omega$ satisfies

$$\tilde{K} = \left\{ x \in \Omega \,\middle|\, \inf_{y \in K} |x - y| \leq \frac{d}{2} \right\}.$$

This φ definitely satisfies our requirements since

$$x \in K \Longrightarrow \varphi(x) = \int_{\tilde{K}} \eta_\varepsilon(x - y) \, \mathrm{d}y = 1,$$
$$x \in \Omega^c \Longrightarrow \varphi(x) = \int 0 \, \mathrm{d}y = 0.$$

□

Exercise 2. *Show that finite measure space $M(X, B_X, \mathbb{C})$ is a Banach space under total variation norm.*

Proof. Let $\{\mu_n\}_{n=1}^{\infty} \subset M(X, B_X, \mathbb{R})$ be a sequence of complex Borel measures such that

$$\lim_{m,n \to \infty} \|\mu_m - \mu_n\| = \lim_{m,n \to \infty} |\mu_m - \mu_n|(X) = 0.$$

For every $E \in B_X$, we have

$$|\mu_m(E) - \mu_n(E)| \leq |\mu_m - \mu_n|(E) \leq |\mu_m - \mu_n|(E) \to 0,\ m, n \to \infty.$$

That is to say $\{\mu_n(E)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Therefore, we can define the limit set function “pointwise” by

$$\mu(E) = \lim_{n\to\infty} \mu_n(E).$$

Finally, we need to verify that μ is a Borel measure in deed. Actually, it is obvious that

$$\mu(\varnothing) = \lim_{n\to\infty} \mu_n(\varnothing) = 0.$$

Moreover, let $\{E_k\}_{k=1}^\infty$ be a disjoint sequence of Borel measurable sets, then

$$\mu\left(\bigsqcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} \mu_n\left(\bigsqcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} \sum_{k=1}^\infty \mu_n(E_k) = \sum_{k=1}^\infty \mu(E_k).$$

Here it is reasonable to swap two limits since the finiteness of μ_n implies the series is absolutely convergent.

We have proved that μ is a measure, and by definition Borel sets are μ-measurable and μ is finite. As a result, we conclude that μ is a finite Borel measure and thus $M(X, B_X, \mathbb{C})$ is a Banach space. □

Exercise 3. *If X is a Banach space, then F is precompact if and only if $\forall\, \varepsilon > 0$, there exists a precompact K_ε such that*

$$F \subset K_\varepsilon + B_\varepsilon(0) = \{f + g \mid f \in K_\varepsilon, g \in B_\varepsilon(0)\}.$$

Proof. $\Longrightarrow$:

Take $K_\varepsilon = F$.

$\Longleftarrow$:

If F is not precompact, then there exist a sequence $\{\varphi_n\}_{n=1}^\infty \subset F$ such that every subsequence of it is not Cauchy.

Consider decomposition $\varphi_n = f_n + g_n$ where $f_n \in K_\varepsilon, g_n \in B_\varepsilon(0)$ for a pending $\varepsilon > 0$, then

$$\|\varphi_m - \varphi_n\| \leq \|f_m - f_n\| + \|g_m - g_n\| \leq \|f_m - f_n\| + 2\varepsilon.$$

Since K_ε is precompact, there is a sufficient large N and a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that

$$\|f_{n_j} - f_{n_k}\| < \varepsilon,\ \forall\, n_j, n_k > N.$$

However, our assumption implies $\{\varphi_{n_k}\}_{n=1}^{\infty}$ is not Cauchy, hence there exist $\varepsilon_0 \geq 0$ and $j, k > N$ such that

$$\|\varphi_{n_j} - \varphi_{n_k}\| \geq \varepsilon_0.$$

Take $\varepsilon = \frac{\varepsilon_0}{3}$ and we obtain

$$\varepsilon_0 \leq \|\varphi_{n_j} - \varphi_{n_k}\| \leq \|f_{n_j} - f_{n_k}\| + 2\varepsilon < 3\varepsilon = \varepsilon_0,$$

a contradiction. □