Solutions to Homework 03

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Folland. ***Real Analysis***

Exercise 1.4.17

Proof. The σ-subadditivity of μ^* indicates

$$\mu^*\left(E \cap \left(\bigsqcup_{k=1}^{\infty} A_k\right)\right) = \mu^*\left(\bigsqcup_{k=1}^{\infty} (E \cap A_k)\right) \leq \sum_{k=1}^{\infty} \mu^*(E \cap A_k),$$

hence we only need to prove the reverse inequality.

According to Carathéodory criterion, we have for any $n > 0$ that

$$\begin{aligned}
\mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n} A_k\right)\right) &= \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n} A_k\right) \cap A_n\right) + \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n} A_k\right) \cap A_n^c\right) \\
&= \mu^*\left(E \cap A_n\right) + \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n} A_k\right) \cap A_n^c\right) \\
&= \mu^*\left(E \cap A_n\right) + \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n-1} A_k\right)\right) \\
&= \mu^*\left(E \cap A_n\right) + \mu^*\left(E \cap A_{n-1}\right) + \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{n-2} A_k\right)\right) \\
&= \cdots\cdots \\
&= \sum_{k=2}^{\infty} \mu^*\left(E \cap A_k\right) + \mu^*\left(E \cap \left(\bigsqcup_{k=1}^{1} A_k\right)\right) \\
&= \sum_{k=1}^{\infty} \mu^*\left(E \cap A_k\right).
\end{aligned}$$

As a consequence,

$$\mu^*\left(E\cap\left(\bigsqcup_{k=1}^{\infty}A_k\right)\right)\geq\mu^*\left(\bigsqcup_{k=1}^{n}(E\cap A_k)\right)=\sum_{k=1}^{n}\mu^*(E\cap A_k).$$

Let $n\to\infty$, and we reach the conclusion.

□

Exercise 1.4.23

(1)

Proof. *(The statement of the problem contains a mistake $-\infty\leq a<b\leq+\infty$, which should be revised as $-\infty\leq a\leq b\leq+\infty$.)*

By **proposition 1.7**, we only need to check

$$\mathcal{E}=\left\{(a,b]\cap\mathbb{Q}\mid a\leq b,a,b\in\overline{\mathbb{R}}\right\}$$

is an **elementary family**.

By choosing $a=b$, we notice $\varnothing\in\mathcal{E}$.

Given $A,B\in\mathcal{E}$, we have $A\cap B=\varnothing\in\mathcal{E}$ if they are disjoint and $A\cap B=A\in\mathcal{E}$ if $A\subset B$. The remaining case is $A=(a_1,a_2]\cap\mathcal{Q}$ and $B(b_1,b_2]\cap\mathcal{Q}$ such that

$$a_1<b_1<a_2<b_2\text{ or }b_1<a_1<b_2<a_2.$$

Without loss of generality, we only focus on the former case, in which

$$A\cap B=(b_1,a_2]\in\mathcal{E}.$$

For $\varnothing\in\mathcal{E}$, its complement in $\mathbb{Q}$ is

$$\varnothing^c=\mathbb{Q}=(-\infty,+\infty]\cap\mathbb{Q}\in\mathcal{E}.$$

For nonempty $A=(a,b]\cap\mathbb{Q}\in\mathcal{E}$, its complement in $\mathbb{Q}$ is

$$A^c=((-\infty,a]\cap\mathbb{Q})\cup((b,+\infty)\cap\mathbb{Q})=((-\infty,a]\cap\mathbb{Q})\cup((b,+\infty]\cap\mathbb{Q}),$$

a finite union of elements in $\mathcal{E}$. □

(2)

Proof. Let

$$A_k=(a_k,b_k]\cap\mathbb{Q},\ \ k=1,2,\cdots$$

be countably many sets in $\mathcal{A}$, then

$$\bigcup_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty}(a_k, b_k]\right) \cap \mathbb{Q} \subset \mathbb{R} \cap \mathbb{Q} = \mathbb{Q}.$$

That is to say, the σ-algebra generated be $\mathcal{A}$ is a subset of $\mathcal{P}(\mathbb{Q})$.

On the other hand, fix an arbitrary $Q \in \mathcal{P}(\mathbb{Q})$. $\mathbb{Q}$ is obviously countable, and $\forall\, q \in Q$, we have

$$\{q\} = \bigcap_{k=1}^{\infty} \left(\left(q - \frac{1}{k}, q\right] \cap Q\right).$$

Therefore, Q is generated by $\mathcal{A}$ through countable union and intersection. $\square$

(3)

Proof. For $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that their union belongs to $\mathcal{A}$, we have

$$\mu_0\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu_0(\varnothing) = 0 = \sum_{k=1}^{\infty} \mu_0(A_k),$$

if $A_1 = A_2 = \cdots = \varnothing$. Otherwise,

$$\mu_0\left(\bigcup_{k=1}^{\infty} A_k\right) = +\infty = \sum_{k=1}^{\infty} \mu_0(A_k).$$

Therefore, μ_0 is definitely a premeasure.

Consider a natural extension μ_1 such that

$$\mu_1(A) = \begin{cases} 0, & A = \varnothing, \\ +\infty, & A \neq \varnothing. \end{cases}$$

It is easy to check μ_1 is a measure on $\mathcal{P}(\mathbb{Q})$.

Let μ_2 be the counting measure such that

$$\mu_2(A) = \begin{cases} |A|, & |A| < +\infty, \\ +\infty, & |A| = +\infty. \end{cases}$$

For nonempty $A \in \mathcal{A}$, A includes all rational numbers in an interval, which are countably infinite. Therefore, $\mu_2(A) = +\infty = \mu_0(A)$.

To conclude, μ_1 and μ_2 are two different measures that coincide on $\mathcal{A}$. $\square$