Solutions to Homework 07

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Folland. *Real Analysis*

Exercise 6.3.27

Proof. Set a (-1)-homogeneous function $K(x,y) = \frac{1}{x+y}$. Comparison Discrimination implies

$$\int_0^{+\infty} |K(1,y)| y^{-\frac{1}{p}} \, \mathrm{d}y = \int_0^{+\infty} \frac{y^{-\frac{1}{p}}}{1+y} \, \mathrm{d}y = C_p < +\infty.$$

Hence T is strong type (p,p) and

$$\|Tf\|_p \leq C_p \|f\|_p.$$

Moreover, Euler's reflection formula implies

$$C_p = \pi \csc \frac{\pi}{p}.$$

□

Exercise 6.3.29

Proof. Set a (-1)-homogeneous function $K(x,y) = x^{\beta-1} y^{-\beta} \chi_{(0,+\infty)}(y-x)$ with, which satisfies

$$\int_0^{+\infty} |K(1,y)| y^{-\frac{1}{p}} \, \mathrm{d}y = \int_1^{+\infty} x^{\beta-1-\frac{1}{p}} = \frac{1}{1-\beta p} < +\infty, \ \beta < \frac{1}{p}.$$

For $f(x) = x^{\gamma} h(x)$, we have

$$\|Tf\|_p \leq \frac{1}{1-\beta p} \|f\|_p, \ Tf(x) = \int_0^{+\infty} K(x,y) f(y) \, \mathrm{d}y.$$

By **Theorem 6.20**, we have

$$\int_0^{+\infty} \left(\int_0^{+\infty} x^{\beta-1} y^{\gamma-\beta} h(y) \chi_{(0,+\infty)}(y-x) \, \mathrm{d}y \right)^p \mathrm{d}x \le \frac{1}{(1-\beta p)^p} \int_0^{+\infty} x^{\gamma p} (h(x))^p \, \mathrm{d}x,$$

which implies

$$\int_0^{+\infty} x^{p(\beta-1)} \left(\int_x^{+\infty} y^{\gamma-\beta} h(y) \, \mathrm{d}y \right)^p \mathrm{d}x \le \frac{1}{(1-\beta p)^p} \int_0^{+\infty} x^{\gamma p} (h(x))^p \, \mathrm{d}x.$$

Let $\beta = \gamma = 1 + \frac{r-1}{p}$, and we obtain one of the inequalities. Swapping x and y in K, we similarly achieve the other inequality by taking $\beta = \frac{r+1}{p}$ and $\gamma = 1 - \frac{r+1}{p}$. □

Exercise 6.4.36

Proof. For $q < p$, we have

$$\begin{aligned} \int |f|^q \, \mathrm{d}\mu &= \int_0^{+\infty} q\lambda^{q-1} \mu\left(\{|f| > \lambda\}\right) \mathrm{d}\lambda \\ &\le q \int_0^1 \lambda^{q-1} \mu\left(\{f \ne 0\}\right) \mathrm{d}\lambda + q\|f\|_{p,\infty}^p \int_1^{+\infty} \lambda^{q-p-1} \, \mathrm{d}\lambda \\ &= \mu\left(\{f \ne 0\}\right) + \frac{q}{p-q} \|f\|_{p,\infty}^p \\ &< +\infty. \end{aligned}$$

For $q > p$, let $\|f\|_\infty = M < +\infty$, then

$$\begin{aligned} \int |f|^q \, \mathrm{d}\mu &= \int_0^M q\lambda^{q-1} \mu\left(\{|f| > \lambda\}\right) \mathrm{d}\lambda \\ &\le q\|f\|_{p,\infty}^p \int_0^M \lambda^{q-p-1} \, \mathrm{d}\lambda \\ &= \frac{q}{q-p} M^{q-p} \\ &< +\infty. \end{aligned}$$

□

Exercise 6.5.43

Proof. First, we compute $A_r f(x)$. Due to the symmetry of $\chi_{[0,1]}$, we assume $x \geq \frac{1}{2}$.

$$\frac{1}{2} \leq x < 1 \Longrightarrow A_r f(x) = \begin{cases} 1, & 0 < r < 1 - x \\ \frac{r+1-x}{2r}, & 1 - x \leq r \leq x \\ \frac{1}{2r}, & r > x \end{cases},$$

$$x = 1 \Longrightarrow A_r f(x) = \begin{cases} \frac{1}{2}, & 0 < r < 1 \\ \frac{1}{2r}, & r \geq 1 \end{cases},$$

$$x > 1 \Longrightarrow A_r f(x) = \begin{cases} 0, & 0 < r < x - 1 \\ \frac{x+1-r}{2r}, & x - 1 \leq r \leq x \\ \frac{1}{2r}, & r > x \end{cases}.$$

Therefore, we obtain the explicit expression

$$Hf(x) = \begin{cases} 1, & x \in (0, 1), \\ \frac{1}{1+|2x-1|}, & x \in (-\infty, 0) \cup (1, +\infty) \end{cases}$$

It is obvious that

$$\|Hf\|_p^p = \int_0^1 1 + 2\int_1^{+\infty} \left(\frac{1}{1 + |2x - 1|}\right)^p = 1 + \int_1^{+\infty} \frac{2^{1-p}}{x^p} = \begin{cases} +\infty, & p = 1, \\ 1 + \frac{2^{1-p}}{p-1}, & p > 1. \end{cases}$$

In the meantime, we have

$$\begin{aligned} \|Hf\|_{1,\infty} &= \sup_{\lambda > 0} \lambda\mu\left(\{|f| > \lambda\}\right) \\ &= \max\left\{ \sup_{\frac{1}{2} \leq \lambda \leq 1} \lambda, \sup_{0 \leq \lambda \leq \frac{1}{2}} \lambda\left(1 + \mu\left(\left\{\frac{1}{x-1} > \lambda\right\}\right)\right)\right\} \\ &= \max\left\{1, \sup_{0 \leq \lambda \leq \frac{1}{2}} (\lambda + 1)\right\} = 1. \end{aligned}$$

□

Exercise 6.5.45

Proof. Let $q = \frac{n}{\alpha}$ and $K(x, y) = |x - y|^{-\alpha}$, then

$$\lambda^q m\left(\{x \mid K(x, y) > \lambda\}\right) = \lambda^q m\left(\{x \mid |x - y| < \lambda^{-\frac{1}{\alpha}}\}\right) = \lambda^q m\left(B_{\lambda^{-\frac{1}{\alpha}}}(y)\right) = C_n.$$

Here C_n equals the volume of n-dimensional unit ball, a constant that only relies on n. Therefore, $K(x, \cdot) \in L^{q,\infty}$, and similar arguments imply $K(\cdot, y) \in L^{q,\infty}$.

According to **Theorem 6.36**, T_α is weak type $(1, \frac{n}{\alpha})$ and strong type (p, r). □