第四教学周9.26

1 Measures

1.5 Borel measures

In case X is a topological space, an outer measure μ on X is said to be Borel-regular if all Borel sets are μ-measurable and if for each subset $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. (Notice that this does not imply $\mu(B \backslash A) = 0$ unless A is μ-measurable and $\mu(A) < \infty$.)

定义1.1. *We say a Borel regular measure μ on a topological space X is "open σ-finite" if $X = \cup_j V_j$ where V_j is open in X and $\mu(V_j) < \infty$ for each $j = 1, 2, \ldots$*

定理1.2. *Suppose X is a topological space with the property that every closed subset of X is the countable intersection of open sets (this trivially bolds e.g. if X is a metric space), suppose μ is an open σ-finite Borel-regular measure on X. Then*

(1)
$$\mu(A) = \inf_{U \text{ open},\, U \supset A} \mu(U)$$

for each subset $A \subset X$, and

(2)
$$\mu(A) = \sup_{C \text{ closed},\, C \subset A} \mu(C)$$

for each μ-measurable subset $A \subset X$.

注记1.1. *In case X is a Hausdorff space (so compact sets in X are closed) which is σ-compact (i.e. $X = \cup_j K_j$ with K_j compact for each j), then the conclusion (2) in the above theorem guarantees that*

$$\mu(A) = \sup_{K \text{ compact},\, K \subset A} \mu(K)$$

for each μ-measurable subset $A \subset X$ with $\mu(A) < \infty$, because under the above conditions on X any closed set C can be written as the union of an increasing sequence of compact sets.

2 Integration

2.1 Measurable functions

定义1.3. *A mapping $f : X \mapsto Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable, or just measurable, if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.*

引理1.4. *Let $\mathcal{C} \subset \mathcal{P}(X)$.*

$$\sigma(f^{-1}\mathcal{C}) = f^{-1}\sigma(\mathcal{C}). \tag{1.1}$$

Proof. Let

$$\mathcal{G} = \left\{B \in \sigma(\mathcal{C}) | f^{-1}B \in \sigma(f^{-1}\mathcal{C})\right\}. \tag{1.2}$$

$\mathcal{C} \subset \mathcal{G}$ and $\mathcal{G}$ is a σ-algebra. Then we conclude that $\mathcal{G} = \sigma(\mathcal{C})$, which shows that

$$f^{-1}\sigma(\mathcal{C}) \subset \sigma(f^{-1}\mathcal{C}).$$

On the other hand,

$$f^{-1}\mathcal{C} \subset f^{-1}\sigma(\mathcal{C}) \Rightarrow \sigma(f^{-1}\mathcal{C}) \subset f^{-1}\sigma(\mathcal{C}). \tag{1.3}$$

□

引理1.5 (measurable functions)**.** *Let f be a mapping between measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, and let $\mathcal{C} \subset \mathcal{P}(T)$ with $\sigma(\mathcal{C}) = \mathcal{T}$. Then*

$$f \text{ is } \mathcal{S}/\mathcal{T}\text{-measurable } \Leftrightarrow f^{-1}\mathcal{C} \subset \mathcal{S}.$$

引理1.6 (composition)**.** *For maps $f : S \to T$ and $g : T \to U$ between the measurable spaces $(S, \mathcal{S}), (T, \mathcal{T}), (U, \mathcal{U})$, we have*

$$f, g \text{ are measurable } \Rightarrow h = g \circ f \text{ is } \mathcal{S}/\mathcal{U}\text{-measurable.}$$

The class of measurable functions is closed under countable limiting operations:

引理1.7 (bounds and limits)**.** *If the functions $f_1, f_2, \ldots : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ are measurable, then so are the functions*

$$\sup_n f_n, \quad \inf_n f_n, \quad \limsup_{n\to\infty} f_n, \quad \liminf_{n\to\infty} f_n.$$

引理1.8 (elementary operations)**.** *If the functions $f, g : (X, \mathcal{M}) \to \mathbb{R}$ are measurable, then so are the functions*

(i) *fg and $af + bg, a, b \in \mathbb{R}$,*

(ii) *f/g when $g \neq 0$ on X.*

引理1.9 (simple approximation)**.** *For any measurable function $f \geq 0$ on $(X, \mathcal{M})$, there exist some simple, measurable functions $f_1, f_2, \ldots : X \to \mathbb{R}^+$ with $f_n \uparrow f$.*

2.2-2.3 Integration

- $\mathcal{S}^+$: the set of non-negative simple measurable functions.
- $\overline{\mathcal{L}}$: the space of all measurable functions $f : X \mapsto \overline{R}$.

定义1.10. *Let $f = \sum_{i=1}^n a_i I_{A_i} \in \mathcal{S}^+$, where $a_i \in \mathbb{R}^+$, $A_i \in \mathcal{M}$. Define*

$$\int_\Omega f d\mu = \sum_{i=1}^n a_i \mu\left(A_i\right). \tag{1.4}$$

命题1.11. *Let $f_n, g_n, f, g \in \mathcal{S}^+$.*

(1) $\mu\left(I_A\right) = \mu(A), \forall A \in \mathcal{M}$;

(2) $\mu(\alpha f) = \alpha\mu(f), \forall \alpha \in \mathbb{R}^+$;

(3) $\mu(f+g) = \mu(f) + \mu(g)$;

(4) $f \leq g \Rightarrow \mu(f) \leq \mu(g)$;

(5) $f_n \downarrow f, \mu\left(f_1\right) < \infty \Rightarrow \mu\left(f_n\right) \downarrow \mu(f)$;

(6) $f_n \uparrow f \Rightarrow \mu\left(f_n\right) \uparrow \mu(f)$;

(7) $f_n \uparrow, g_n \uparrow, \lim_{n\to\infty} f_n \leq \lim_{n\to\infty} g_n \Rightarrow \lim_{n\to\infty} \mu\left(f_n\right) \leq \lim_{n\to\infty} \mu\left(g_n\right)$.

定义1.12. *To extend the integral to general measurable functions $f \geq 0$, we choose some simple measurable functions $f_1, f_2, ...$ with $0 \leq f_n \uparrow f$, and define*

$$\mu(f) = \lim_n \mu(f_n).$$

We need to show that the limit is independent of the choice of approximating sequence (f_n).

引理1.13 (consistence)**.** *Let $f, f_1, f_2, \ldots$ and g be measurable functions on a measure space $(X, \mathcal{M}, \mu)$, where all but f are simple. Then*

$$\left.\begin{array}{l} 0 \leq f_n \uparrow f \\ 0 \leq g \leq f \end{array}\right\} \Rightarrow \mu(g) \leq \lim_{n\to\infty} \mu(f_n).$$

定理1.14 (monotone convergence, Levi)**.** *For any measurable functions $f, f_1, f_2 \ldots$ on $(X, \mathcal{M}, \mu)$, we have*

$$0 \leq f_n \uparrow f \quad \Rightarrow \quad \mu f_n \uparrow \mu f$$

引理1.15 (Fatou)**.** *For any measurable functions $f_1, f_2, \ldots \geq 0$ on $(S, \mathcal{M}, \mu)$, we have*

$$\liminf_{n\to\infty} \mu f_n \geq \mu \liminf_{n\to\infty} f_n.$$

定理1.16 (extended dominated convergence, Lebesgue)**.** *Let $f, f_1, f_2, \dots$ and $g, g_1, g_2, \dots \geq 0$ be measurable functions on $(X, \mathcal{M}, \mu)$. Then*

$$\left.\begin{array}{c} f_n \to f \\ |f_n| \leq g_n \to g \\ \mu g_n \to \mu g < \infty \end{array}\right\} \Rightarrow \mu f_n \to \mu f.$$

2.4 Modes of convergence

定义1.17. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $(f_n)_{n \geq 1}$ be real-valued $\mathcal{M}$-measurable function on X.*

- *$\{f_n\}$ converges to f* ***almost uniformly*** *if for $\forall \varepsilon > 0$, $\exists N \in \mathcal{M}$, $\mu(N) < \varepsilon$ such that*

$$\lim_{n \to \infty} \sup_{x \in N^c} |f_n(x) - f(x)| = 0. \tag{1.5}$$

定理1.18. *Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $(f_n)_{n \geq 1}$ be real-valued $\mathcal{M}$-measurable function on X.*

- *$f_n \overset{a.e.}{\to} f$, if and only if $\forall \varepsilon > 0$*

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} [|f_i - f| \geq \varepsilon]\right) = 0. \tag{1.6}$$

- *$f_n \overset{a.un.}{\to} f$, if and only if $\forall \varepsilon > 0$*

$$\lim_{n \to \infty} \mu\left(\bigcup_{i=n}^{\infty} [|f_n - f| \geq \varepsilon]\right) = 0. \tag{1.7}$$

- *$f_n \overset{\mu}{\to} f$, if and only if for any subsequence $(f_{n'})$, there exists its subsequence $f_{n'_k}$ such that*

$$f_{n'_k} \overset{a.un.}{\to} f. \tag{1.8}$$

定理1.19. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

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$$f_n \xrightarrow{a.un.} f \Rightarrow f_n \xrightarrow{a.e.} f; \quad f_n \xrightarrow{a.un.} f \Rightarrow f_n \overset{\mu}{\to} f. \tag{1.9}$$

- *If μ is finite, then*

$$f_n \xrightarrow{a.e} f \Leftrightarrow f_n \xrightarrow{a.un.} f. \tag{1.10}$$

- *If $f_n \overset{\mu}{\to} f$, there exists a subsequence f_{n_k} such that $f_{n_k} \overset{a.e.}{\to} f$.*

2.5 Product measures

Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ be two measure spaces. Define

$$\mathcal{C} := \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}, \tag{1.11}$$
$$\pi(A \times B) := \mu(A)\nu(B). \tag{1.12}$$

命题1.20. *π is σ-additive on $\mathcal{C}$.*

If $E \subset X \times Y$, for $x \in X$ and $y \in Y$ we define the x-section E_x and the y-section E^y of E by

$$E_x = \{y \in Y : (x, y) \in E\}, \quad E^y = \{x \in X : (x, y) \in E\}.$$

Also, if f is a function on $X \times Y$ we define the x-section f_x and the y-section f^y of f by

$$f_x(y) = f^y(x) = f(x, y)$$

命题1.21. *a. If $E \in \mathcal{M} \times \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.*

b. If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.

定理1.22. *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X and Y, respectively, and*

$$\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y).$$

定理1.23 (Fubini-Tonelli Theorem)**.** *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma-$finite measure spaces.*

a. (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\int f d(\mu \times \nu) = \int \left[\int f(x, y) d\nu(y)\right] d\mu(x) = \int \left[\int f(x, y) d\mu(x)\right] d\nu(y). \tag{1.13}$$

b. (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x d\nu$ and $h(x) = \int f^y d\nu$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively, and (1.13) holds.

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