# 1 第14教学周12.5

## Applications

**引理1.1** (Product lemma). *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, $s \geq 0$. Then we have*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}. \tag{1.1}$$

**命题1.2** (Hardy's inequality). *For $f \in \mathcal{S}(\mathbb{R}^d)$, and $s \in [0, \frac{d}{2})$,*

$$\left\|\frac{f}{|x|^s}\right\|_{L^2} \lesssim \||\nabla|^s f\|_{L^2}. \tag{1.2}$$

## 1.6 Method of stationary and non-stationary phase

**引理1.3.** *If $\nabla\phi \neq 0$ on $\operatorname{supp} a$, then the integral*

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(\xi)} a(\xi) d\xi\right| \leq C(N, a, \phi)\lambda^{-N}, \quad \lambda \to \infty \tag{1.3}$$

*for arbitrary $N \geq 1$.*

**引理1.4.** *If $\nabla\phi(\xi_0) = 0$ for some $\xi_0 \in \operatorname{supp}(a), \nabla\phi \neq 0$ away from $\xi_0$, and the Hessian of $\phi$ at the stationary point $\xi_0$ is nondegenerate, i.e., $\det \nabla^2\phi(\xi_0) \neq 0$, then for all $\lambda \geq 1$*

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(\xi)} a(\xi) d\xi\right| \leq C(d, a, \phi)\lambda^{-d/2}. \tag{1.4}$$

# Elementary of distribution theory

## 1.1 Basic spaces

$\Omega$: open set. $K$: compact set.

**定义1.5** ($\mathcal{E}(\Omega)$ space). *A sequence $\varphi_j \in C^\infty(\Omega)$, $j \in \mathbb{N}$, converges in $C^\infty(\Omega)$ to a function $\varphi \in C^\infty(\Omega)$ as $j \to \infty$, if and only if for any compact set $K \subset \Omega$ and any $m \in \mathbb{N}_0$ one has*

$$\lim_{j\to\infty} \sup_{\alpha\in\mathbb{N}_0^n, |\alpha|\leq m} \sup_{x\in K} |\partial^\alpha(\varphi_j - \varphi)(x)| = 0.$$

**定义1.6** ($\mathcal{D}(\Omega)$ space)**.** *Let* $\varphi_j \in C_c^\infty(\Omega)$, *and* $\varphi \in C_c^\infty(\Omega)$.

$$\varphi_j \to \varphi \quad \text{in } \mathcal{D}(\Omega) \tag{1.5}$$

*if and only if*

1. $\exists K \subseteq \Omega$ *compact set such that* $\operatorname{supp}\varphi_j \subset K$ *for* $j \geq 1$, *and* $\operatorname{supp}\varphi \subset K$.
2. $\forall \alpha$, $\lim_{j\to\infty}\sup_{x\in K}|\partial^\alpha(\varphi_j - \varphi)(x)| = 0$.

**命题1.7.** *1.* $\partial^\alpha : \varphi \mapsto \partial^\alpha\varphi$ *is continuous in* $\mathcal{D}(\Omega)$, $\mathcal{S}(\mathbb{R}^d)$, *and* $\mathcal{E}(\Omega)$.

*2. For* $g \in C^\infty(\Omega)$, $M_g : \varphi \mapsto g\varphi$ *is continuous in* $\mathcal{D}(\Omega)$ *and* $\mathcal{E}(\Omega)$.

---

**作业**

$P_{300}$ **27;** $P_{308}$ **30, 36;**