1 第12教学周11.21

1.3 Regularity and Fourier series

定理1.1. *Let $f \in L^1(\mathbb{T})$ and assume that $\widehat{f}(n) = -\widehat{f}(-n) \geq 0$. Then*

$$\sum_{n>0} \frac{1}{n} \hat{f}(n) < \infty.$$

推论1.2. *If $a_n > 0, \sum a_n/n = \infty$, then $\sum a_n \sin nt$ is not a Fourier series.*

Smoothness and Fourier coefficients

定理1.3. *If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\widehat{f}(n) = o(1/n)$.*

If f is k-times differentiable, and $f^{(k-1)}$ is absolutely continuous, then

$$|\hat{f}(n)| \leq \min_{0 \leq j \leq k} \frac{\left\|f^{(j)}\right\|_{L^1}}{|n|^j}.$$

If f is infinitely differentiable, then

$$|\hat{f}(n)| \leq \min_{0 \leq j} \frac{\left\|f^{(j)}\right\|_{L^1}}{|n|^j}.$$

If $f \in C^\alpha(\mathbb{T})$, then $\hat{f}(n) = O\left(n^{-\alpha}\right)$.

定义1.4. *A function f is analytic on $\mathbb{T}$ if in a neighborhood of every $t_0 \in \mathbb{T}, f(t)$ can be represented by a power series (of the form $\sum_{n=0}^{\infty} a_n \left(t - t_0\right)^n$).*

Show that f is analytic if, and only if, f is infinitely differentiable on $\mathbb{T}$ and there exists a number R such that

$$\sup_t \left|f^{(n)}(t)\right| \leq n! R^n, \quad n > 0.$$

Show that f is analytic on $\mathbb{T}$ if, and only if, there exist constants $K > 0$ and $a > 0$ such that $|\hat{f}(j)| \leq K e^{-a|j|}$.

定义1.5. *Gevrey class. $f \in \mathcal{G}_s$, if there exist constants $K > 0$ and $a > 0$ such that*

$$|\widehat{f}(n)| \leq K e^{-a|n|^{\frac{1}{s}}}. \tag{1.1}$$

H^s **Sobolev space**

定义1.6. *For any* $s \in \mathbb{R}$ *define the Hilbert space* $H^s(\mathbb{T})$ *by means of the norm*

$$\|f\|_{H^s}^2 := |\hat{f}(0)|^2 + \sum_{n\in\mathbb{Z}} |n|^{2s} |\hat{f}(n)|^2.$$

定理1.7. *For any* $1 \geq \alpha > 0$ *one has* $C^\alpha(\mathbb{T}) \hookrightarrow H^\beta(\mathbb{T})$ *for arbitrary* $0 < \beta < \alpha$. *In particular, for any* $f \in C^\alpha(\mathbb{T})$ *with* $\alpha > \frac{1}{2}$ *one has*

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)| < \infty$$

and thus $C^\alpha(\mathbb{T}) \hookrightarrow \mathbb{A}(\mathbb{T})$ *for any* $\alpha > \frac{1}{2}$.

Introduce a norm to $A(\mathbb{T})$ by

$$\|f\|_{A(\mathbb{T})} = \sum_{-\infty}^{\infty} |\hat{f}(n)|.$$

引理1.8. *Assume that* $f, g \in A(\mathbb{T})$. *Then* $fg \in A(\mathbb{T})$ *and*

$$\|fg\|_{A(\mathbb{T})} \leq \|f\|_{A(\mathbb{T})} \|g\|_{A(\mathbb{T})}.$$

1.4 Fourier coefficient of Borel measure

定理1.9. *(Parseval's formula). Let* $f \in C(\mathbb{T}), \mu \in \mathcal{M}(\mathbb{T})$; *then*

$$\langle f, \mu \rangle = \lim_{N\to\infty} \sum_{-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n) \overline{\hat{\mu}(n)}.$$

推论1.10 (Uniqueness theorem)**.** *If* $\hat{\mu}(n) = 0$ *for all* n, *then* $\mu = 0$.

定理1.11. *Let* $\{a_n\}_{n=-\infty}^{\infty}$ *be a sequence of complex numbers. Then the following two conditions are equivalent:*

(a) There exists $\mu \in \mathcal{M}(\mathbb{T}), \|\mu\| \leq C$, *such that* $\hat{\mu}(n) = a_n$ *for all* n.

(b) For all trigonometric polynomials P

$$\left|\sum \hat{P}(n)\overline{a_n}\right| \leq C\|P\|_{C(\mathbb{T})}.$$

推论1.12. *A trigonometric series* $S \sim \sum a_n e^{int}$ *is the Fourier series of some* $\mu \in \mathcal{M}(\mathbb{T}), \|\mu\|_{var} \leq C$, *if, and only if,*

$$\|K_N * S\| \leq C, \quad \forall N.$$

1.5 Higher dimension

命题1.13. *The space of trigonometric polynomials is dense in* $C\left(\mathbb{T}^d\right)$*, and one has Parseval's identity for* $L^2\left(\mathbb{T}^d\right)$*, i.e.,*

$$\|f\|_2^2=\sum_{\nu\in\mathbb{Z}^d}|\hat{f}(\nu)|^2\quad\forall f\in L^2\left(\mathbb{T}^d\right).$$

If $f\in C^\infty(\mathbb{T}^d)$ *then the Fourier series associated with* f *converges uniformly to* f *irrespective of the way in which the partial sums are formed.*

The Fourier transform on $\mathbb{R}^d$

1.1 Basic definitions: Fourier transform, Schwartz space

定义1.14. *Definition 4.2 The Schwartz space* $\mathcal{S}\left(\mathbb{R}^d\right)$ *is defined as the collection of all functions in* $C^\infty\left(\mathbb{R}^d\right)$ *that decay rapidly, together with all derivatives. In other words,* $f\in\mathcal{S}\left(\mathbb{R}^d\right)$ *if and only if* $f\in C^\infty\left(\mathbb{R}^d\right)$ *and*

$$x^\alpha\partial^\beta f(x)\in L^\infty\left(\mathbb{R}^d\right)\quad\forall\alpha,\beta,$$

where α,β *are arbitrary multi-indices. We introduce the following notion of convergence in* $\mathcal{S}\left(\mathbb{R}^d\right)$ *: a sequence* $f_n\in\mathcal{S}\left(\mathbb{R}^d\right)$ *converges to* $g\in\mathcal{S}\left(\mathbb{R}^d\right)$ *if and only if*

$$\left\|x^\alpha\partial^\beta\left(f_n-g\right)\right\|_\infty\to 0,\quad n\to\infty,$$

for all α,β.

命题1.15. *The Fourier transform is a continuous operation from the Schwartz space into itself.*

作业

P_{239} **1, 2;** P_{254} **14;** P_{263} **28, 32;**