1 第13教学周11.28

1.2 The inversion theorem and Plancherel theorem

命题1.1. *For any* $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ *one has*

$$f(x)=\int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi \quad \forall x \in \mathbb{R}^d.$$

Moreover, $\forall f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int \widehat{f} g=\int f \widehat{g}, \tag{1.1}$$

$$\int f\overline{g}=\int \widehat{f}\overline{\widehat{g}}, \tag{1.2}$$

$$\|f\|_{L^2}=\|\widehat{f}\|_{L^2}. \tag{1.3}$$

定义1.2 (Tempered distribution)**.** *The dual of* $\mathcal{S}$*, denoted by* $\mathcal{S}'$*, is the space of tempered distributions. In other words, any* $u \in \mathcal{S}'$ *is a continuous functional on* $\mathcal{S}$*, and we write* $\langle u, \phi\rangle$ *for* u *applied to* $\phi \in \mathcal{S}$*. The space* $\mathcal{S}'$ *is equipped with the weak-* topology. Thus,* $u_n \to u$ *in* $\mathcal{S}'$ *if and only if* $\langle u_n, \phi\rangle \to \langle u, \phi\rangle$ *as* $n \to \infty$ *for every* $\phi \in \mathcal{S}$.

Naturally, $L^p\left(\mathbb{R}^d\right) \hookrightarrow \mathcal{S}'\left(\mathbb{R}^d\right)$ by the rule

$$\langle f, \phi\rangle=\int_{\mathbb{R}^d} f(x)\phi(x)dx, \quad f \in L^p\left(\mathbb{R}^d\right), \phi \in \mathcal{S}\left(\mathbb{R}^d\right).$$

A linear functional u on $\mathcal{S}$ is continuous if and only if there exists an N such that

$$|\langle u, \phi\rangle| \leq C \sum_{|\alpha|,|\beta| \leq N}\left\|x^\alpha \partial^\beta \phi\right\|_{\infty} \quad \forall \phi \in \mathcal{S}.$$

定义1.3. *If* $u \in \mathcal{S}'$*, we define*

$$(\mathcal{F}u)(\varphi)=: u(\mathcal{F}\varphi). \tag{1.4}$$

Fourier transform on L^1+L^2

- $f \in L^1$.

$$(\mathcal{F}f)(x)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx. \tag{1.5}$$

  where $\mathcal{F}f$ is regarded as the generalize Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$.

- $f \in L^2$. Choose $\{f_n\} \subset \mathcal{S}$ such that

$$\lim_{n\to\infty}\|f_n-f\|_{L^2}=0. \tag{1.6}$$

- $f \in L^1+L^2$.

1.3 Poisson summation formula

命题1.4. *For any $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ one has*

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{v \in \mathbb{Z}^d} \hat{f}(v).$$

1.4 Sobolev spaces and inequalities

定义1.5. *Sobolev spaces H^s and $\dot{H}^s$ are defined in terms of the norms*

$$\|f\|_{H^s} = \left\| \langle \xi \rangle^s \hat{f} \right\|_2, \quad \|f\|_{\dot{H}^s} = \left\| |\xi|^s \hat{f} \right\|_2$$

for any $s \in \mathbb{R}$ where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

These spaces are the completion of $\mathcal{S}\left(\mathbb{R}^d\right)$ under the norms. For the homogeneous space $\dot{H}^s\left(\mathbb{R}^d\right)$ one needs the restriction $s > -d/2$ since otherwise $\mathcal{S}$ is not dense in this space.

引理1.6. *For any $f \in H^s\left(\mathbb{R}^d\right)$ one has*

$$\|f\|_p \leq C(s) \|f\|_{H^s\left(\mathbb{R}^d\right)} \quad \forall 2 \leq p \leq \infty,$$

provided that $s > d/2$.

引理1.7 (Trace)**.** *Let $f \in \mathcal{S}$, $x : k$-dimension.*

$$\|f\|_{L_x^\infty L_y^2} \lesssim \|f\|_{H^s}, \quad s > \frac{k}{2}. \tag{1.7}$$

1.5 Littlewood-Paley theory

Let $\chi(\xi) \in C_c^\infty\left(\mathbb{R}^n\right)$ be such that

$$\chi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2. \end{cases}$$

Define

$$\psi(\xi) = \chi(\xi) - \chi(2\xi),$$
$$\psi_j(\xi) = \psi\left(\frac{\xi}{2^j}\right), \quad \chi_j(\xi) = \chi_j(\frac{\xi}{2^j}).$$

Verify that

$$\sum_{j\in\mathbb{Z}}\psi_j(\xi)=1,\forall\xi\neq 0$$

定义1.8. *We define Littlewood-Paley operators as projections on frequency space*

$$\begin{aligned}P_k f&=\mathcal{F}^{-1}\left(\psi_k\hat{f}\right),\\ P_{\le k}f&=\mathcal{F}^{-1}\left(\chi_k\hat{f}\right),\\ P_{>k}f&=\left(I-P_{\le k}\right)f.\end{aligned}$$

$$f\sim\sum_{k\in\mathbb{Z}}P_k f$$

命题1.9. *(1). $P_{\le k}f\overset{\mathcal{S}}{\to}f$, as $k\to\infty$, $\forall f\in\mathcal{S}$.*

(2). $P_{\le k}f\overset{\mathcal{S}'}{\to}f$, as $k\to\infty$, $\forall f\in\mathcal{S}'$.

(3). $p\in[1,\infty)$. $P_{\le k}f\overset{L^p}{\to}f$, as $k\to\infty$, $\forall f\in L^p$.

命题1.10 (Bernstein inequalities)**.** *Let $f\in\mathcal{S}$, $1\le p\le q\le\infty$.*

$$\left\|P_{\le k}f\right\|_q\lesssim 2^{kn\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p \tag{1.8}$$

$$\left\|P_k f\right\|_q\lesssim 2^{kn\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_p \tag{1.9}$$

$$\left\|\nabla P_{\le k}f\right\|_p\lesssim 2^k\|f\|_p \tag{1.10}$$

$$\left\|P_k f\right\|_p\sim 2^{-k}\left\|\nabla P_k f\right\|_p \tag{1.11}$$

$$\left\|P_{>k}f\right\|_p\lesssim 2^{-k}\left\|\nabla f\right\|_p \tag{1.12}$$

命题1.11 (GN inequality)**.**

$$\left\|\partial^i f\right\|_{L^p}\lesssim\|f\|_{L^p}^{1-\frac{i}{m}}\left\|\partial^m f\right\|_{L^p}^{\frac{i}{m}},\quad 0\le i\le m. \tag{1.13}$$

命题1.12 (Sobolev inequality)**.**

$$\|f\|_{L^p}\lesssim\|f\|_{\dot{H}^s} \tag{1.14}$$

where

$$p\in[2,\infty),\quad s\in[0,\frac{d}{2}),\quad\frac{1}{p}=\frac{s}{d}. \tag{1.15}$$

作业

1. 证明P.V. $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$, 计算$\mathcal{F}(\text{P.V. } \frac{1}{x})$.
2. 计算广义Fourier变换
$$e^{i|x|^2}, d \geq 1. \tag{1.16}$$
$$\frac{1}{1+|x|^2}, \quad d=1, d=3. \tag{1.17}$$
3. $s \in (0,1)$, $\chi(x) \in C_c^\infty(\mathbb{R}^d)$. 判断$\mathcal{F}(\chi(x)|x|^s) \in L^1$?
4. $f \in \mathcal{S}(\mathbb{R}^d)$, $\text{supp}(\hat{f}) \subset E \subset \mathbb{R}^d$,其中$E$是Borel可测，证
$$\|f\|_q \leq |E|^{1/p-1/q}\|f\|_p \quad \forall 1 \leq p \leq q \leq \infty.$$
$|E|$是E的Lebesgue测度。