1 第11教学周11.14

1.1 Partial sums and Dirichlet kernel

let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus (in other words, the circle).

The space of continuous functions $C(\mathbb{T})$.

The space of Hölder continuous functions $C^\alpha(\mathbb{T})$ where $0 < \alpha \le 1$.

The Lebesgue spaces $L^p(\mathbb{T})$ where $1 \le p \le \infty$.

The space of complex Borel measures on $\mathbb{T}$ will be denoted by $\mathcal{M}(\mathbb{T})$.

Any $\mu \in \mathcal{M}(\mathbb{T})$ has associated with it a Fourier series

$$\mu \sim \sum_{n=-\infty}^{\infty} \hat{\mu}(n) e(nx)$$

where $e(x) := e^{2\pi i x}$ and

$$\hat{\mu}(n) := \int_0^1 e(-nx)\mu(dx) = \int_{\mathbb{T}} e(-nx)\mu(dx).$$

The symbol $\sim$ is formal and simply means that the series on the righthand side is associated with μ.

If $\mu(dx) = f(x)dx$ where $f \in L^1(\mathbb{T})$, then we may write $\hat{f}(n)$ instead of $\hat{\mu}(n)$.

定义1.1 (Dirichlet kernel)**.** *The partial sums of $f \in L^1(\mathbb{T})$ are defined as*

$$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n) e(nx) = \sum_{n=-N}^{N} \int_{\mathbb{T}} e(-ny) f(y) dy e(nx) \\ &= \int_{\mathbb{T}} \sum_{n=-N}^{N} e(n(x-y)) f(y) dy = \int_{\mathbb{T}} D_N(x-y) f(y) dy \end{aligned}$$

where $D_N(x) := \sum_{n=-N}^{N} e(nx)$ is the Dirichlet kernel. In other words, we have shown that the partial sum operator S_N is given by convolution with the Dirichlet kernel D_N :

$$S_N f(x) = (D_N * f)(x).$$

练习1.2. *Verify that, for each integer $N \ge 0$,*

(1) $\int_{\mathbb{T}} D_N(t) dt = 1$.

(2)

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

(3)

$$|D_N(x)| \le C \min\left(N, \frac{1}{|x|}\right).$$

(4)

$$C^{-1} \log N \le \|D_N\|_{L^1(\mathbb{T})} \le C \log N.$$

引理1.3 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{T})$ then $\hat{f}(n) \to 0$ as $n \to \infty$.*

引理1.4 (Dini's criteria). *If for some x, $\exists \delta > 0$, so that*

$$\int_{|t|<\delta} \left|\frac{f(x+t) - f(x)}{t}\right| dt < \infty, \tag{1.1}$$

then $\lim_{N\to\infty} S_N f(x) = f(x)$.

引理1.5 (Jordan's criteria). *If f is a function of bounded variation in a neighbourhood of x, then then $\lim_{N\to\infty} S_N f(x) = \frac{1}{2}[f(x+0) + f(x-0)]$.*

1.2 Approximate identities and Fejér kernel

Setting

$$K_N := \frac{1}{N} \sum_{n=0}^{N-1} D_n,$$

where K_N is called the Fejér kernel, one therefore has $\sigma_N f = K_N * f$.

练习1.6. *Let K_N be a Fejér kernel with N a positive integer. Verify that*

(1)

$$\widehat{K}_N(n) = \left(1 - \frac{|n|}{N}\right)^+.$$

(2)

$$K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2.$$

(3)

$$0 \le K_N(x) \le C N^{-1} \min\left(N^2, x^{-2}\right).$$

定义1.7. *The family* $\{\Phi_N\}_{N=1}^{\infty} \subset L^{\infty}(\mathrm{T})$ *forms an approximate identity provided that*

(1) $\int_0^1 \Phi_N(x)dx = 1$ *for all* N,

(2) $\sup_N \int_0^1 |\Phi_N(x)|\, dx < \infty$,

(3) *for all* $\delta > 0$ *one has* $\int_{|x|>\delta} |\Phi_N(x)|\, dx \to 0$ *as* $N \to \infty$.

命题1.8. *For any approximate identity* $\{\Phi_N\}_{N=1}^{\infty}$ *one has the following.*

(i) *If* $f \in C(\mathbb{T})$ *then* $\|\Phi_N * f - f\|_\infty \to 0$ *as* $N \to \infty$.

(ii) *If* $f \in L^p(\mathbb{T})$, *where* $1 \le p < \infty$, *then* $\|\Phi_N * f - f\|_p \to 0$ *as* $N \to \infty$.

(iii) *For any measure* $\mu \in \mathcal{M}(\mathbb{T})$, *one has*

$$\Phi_N * \mu \rightharpoonup \mu, \quad N \to \infty,$$

in the weak- sense.*

推论1.9. *The exponential family* $\{e(nx)\}_{n\in\mathbb{Z}}$ *satisfies the following properties.*

(i) *The trigonometric polynomials are dense in* $C(\mathbb{T})$ *in the uniform topology and in* $L^p(\mathbb{T})$ *for any* $1 \le p < \infty$.

(ii) *For any* $f \in L^2(\mathbb{T})$,

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

(iii) *The exponentials* $\{e(nx)\}_{n\in\mathbb{Z}}$ *form an orthonormal basis in* $L^2(\mathbb{T})$.

(iv) *For all* $f, g \in L^2(\mathbb{T})$ *one has Parseval's identity,*

$$\int_{\mathbb{T}} f(x)\bar{g}(x)dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)}.$$

Failure of convergence.

定理1.10. *There exists a continuous function whose Fourier series diverges at a point.*

命题1.11. *The following statements are equivalent for any* $1 \le p \le \infty$:

(i) *for every* $f \in L^p(\mathbb{T})$ (*or* $f \in C(\mathbb{T})$ *if* $p = \infty$) *one has*

$$\|S_N f - f\|_p \to 0 \text{ as } N \to \infty;$$

(ii) $\sup_N \|S_N\|_{p\to p} < \infty$.

推论1.12. *Fourier series do not converge on* $C(\mathbb{T})$ *and* $L^1(\mathbb{T})$, *i.e., there exists* $f \in C(\mathbb{T})$ *such that* $\|S_N f - f\|_\infty \nrightarrow 0$ *and* $g \in L^1(\mathbb{T})$ *such that* $\|S_N g - g\|_1 \nrightarrow 0$ *as* $n \to \infty$.

1.3 Regularity and Fourier series

命题1.13 (Bernstein). *Let f be a trigonometric polynomial with $\hat{f}(k) = 0$ for all $|k| > n$. Then*

$$\|f'\|_p \leq Cn\|f\|_p$$

for any $1 \leq p \leq \infty$. The constant C is absolute.

引理1.14. *Let $\{a_n\}_{n\in Z}$ be an even sequence of nonnegative numbers that tend to zero, which is convex in the following sense:*

$$a_{n+1} + a_{n-1} - 2a_n \geq 0 \quad \forall n > 0.$$

Then there exists $f \in L^1(\mathbb{T})$ with $f \geq 0$ and $\hat{f}(n) = a_n$.

推论1.15. *Suppose that $f \in L^1(\mathbb{T})$ satisfies $\hat{f}(j) = 0$ for all j with $|j| < n$. Then*

$$\|f''\|_p \geq Cn^2\|f\|_p$$

holds for all $1 \leq p \leq \infty$.

作业

P_{269} **33, 34, 35, 36;**