第六教学周10.10

3.2 The Lebesgue-Radon-Nikodym theorem

注记1.13.1. *The decomposition* $\nu = \lambda + \rho$ *where* $\lambda \perp \mu$ *and* $\rho \ll \mu$ *is called the Lebesgue decomposition of* ν *with respect to* μ*.*

注记1.13.2. *In the case where* $\nu \ll \mu$*, we have that* $d\nu = f d\mu$ *for some* f*. This result is usually known as the Radon-Nikodym theorem, and* f *is called the Radon-Nikodym derivative of* ν *with respect to* μ*. We denote it by* $d\nu/d\mu$ *:*

$$d\nu = \frac{d\nu}{d\mu} d\mu$$

命题1.14. *Suppose that* ν *is a* σ*-finite signed measure and* μ, λ *are* σ*-finite measures on* $(X, \mathcal{M})$ *such that* $\nu \ll \mu$ *and* $\mu \ll \lambda$*.*

a. If $g \in L^1(\nu)$*, then* $g(d\nu/d\mu) \in L^1(\mu)$ *and*

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

b. We have $\nu \ll \lambda$*, and*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \quad \lambda\text{-a.e.}$$

3.3 Complex measures

定义1.15. *A complex measure on a measurable space* $(X, \mathcal{M})$ *is a map* $\nu : \mathcal{M} \to \mathbb{C}$ *such that*

- $\nu(\varnothing) = 0$;
- *if* $\{E_j\}$ *is a sequence of disjoint sets in* $\mathcal{M}$*, then* $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu(E_j)$*, where the series converges absolutely.*

In particular, infinite values are not allowed, so a positive measure is a complex measure only if it is finite. Example: If μ is a positive measure and $f \in L^1(\mu)$, then $f d\mu$ is a complex measure.

定理1.16 (Lebesgue-Radon-Nikodym theorem)**.** *If* ν *is a complex measure and* μ *is a* σ*-finite positive measure on* $(X, \mathcal{M})$*, there exist a complex measure* λ *and an* $f \in L^1(\mu)$ *such that* $\lambda \perp \mu$ *and* $d\nu = d\lambda + f d\mu$*. If also* $\lambda' \perp \mu$ *and* $d\nu = d\lambda' + f' d\mu$*, then* $\lambda = \lambda'$ *and* $f = f'$ μ*-a.e.*

3.4 Differentiation on Euclidean space

引理1.17 (Elementary version: Vitali covering)**.** *Suppose $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^n$. Then there exists a disjoint sub-collection $\{B_{j_i}\}_{i=1}^M$ of $\mathcal{B}$ that satisfies*

$$\bigcup_{l=1}^{N} B_l \subset \bigcup_{i=1}^{M} 3B_{j_i}. \tag{1.1}$$

定义1.18. *A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called locally integrable (with respect to Lebesgue measure) if $\int_K |f(x)|dx < \infty$ for every bounded measurable set $K \subset \mathbb{R}^n$.*

We denote the space of locally integrable functions by L^1_{loc}. If $f \in L^1_{loc}$, $x \in \mathbb{R}^n$, and $r > 0$, we define $A_r f(x)$ to be the average value of f on $B(r, x)$:

$$A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy.$$

If $f \in L^1_{loc}$, we define its Hardy-Littlewood maximal function Hf by

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|dy.$$

定理1.19 (The maximal theorem)**.** *There is a constant $C > 0$ such that for all $f \in L^1$ and all $\alpha > 0$,*

$$m(\{x : Hf(x) > \alpha\}) \le \frac{C}{\alpha} \int |f(x)|dx.$$

定理1.20. *If $f \in L^1_{\mathrm{loc}}$, then $\lim_{r\to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.*

This result can be rephrased as follows: If $f \in L^1_{\mathrm{loc}}$,

$$\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} [f(y) - f(x)]dy = 0 \text{ for a.e. } x.$$

Actually, something stronger is true. Let us define the Lebesgue set L_f of f to be

$$L_f = \left\{ x : \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|dy = 0 \right\}.$$

定理1.21. *If $f \in L^1_{\mathrm{loc}}$, then $m\left((L_f)^c\right) = 0$.*

4.1 Topological spaces

Let X be a nonempty set.

定义1.22 (Topological Space)**.** *A topology on X is a family $\mathcal{T}$ of subsets of X that satisfies the following conditions:*

- $\emptyset$, $X \in \mathcal{T}$.
- *Closed under finite intersections:* $U, V \in \mathcal{T} \Longrightarrow U \cap V \in \mathcal{T}$.
- *Closed under arbitrary unions:* $\{U_i\}_{i\in I} \subset \mathcal{T} \Longrightarrow \bigcup_{i\in I} U_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space.

If X is any nonempty set, $\mathcal{P}(X)$ and $\{\varnothing, X\}$ are topologies on X. They are called the discrete topology and the trivial (or indiscrete) topology, respectively.

If X is a metric space, the collection of all open sets with respect to the metric is a topology on X.

5.1 Normed vector spaces

Let K denote either $\mathbb{R}$ or $\mathbb{C}$, and let X be a vector space over K.

定义1.23. *A seminorm on X is a function $x \mapsto \|x\|$ from X to $[0, \infty)$ such that*

- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$ *(the triangle inequality),*
- $\|\lambda x\| = |\lambda|\|x\|$ *for all* $x \in X$ *and* $\lambda \in K$.

The second property clearly implies that $\|0\| = 0$. A seminorm such that $\|x\| = 0$ only when $x = 0$ is called a norm, and a vector space equipped with a norm is called a normed vector space (or normed linear space).

定义1.24 (Banach space)**.** *A normed vector space that is complete with respect to the norm metric is called a Banach space.*

6.1 Basic theory of L^p spaces

- Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Let $(X, \mathcal{M}, \mu)$ denote a measure space.

定义1.25 (L^p spaces)**.** *Let $1 < p < \infty$; If f is a measurable function on X, we define*

$$\|f\|_p = \left[\int |f|^p d\mu\right]^{1/p}$$

(allowing the possibility that $\|f\|_p = \infty$), and we define

$$L^p(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty\}$$

We abbreviate $L^p(X, \mathcal{M}, \mu)$ by $L^p(\mu), L^p(X)$, or simply L^p when this will cause no confusion.

定义1.26. *We set*

$$L^\infty(X) = \left\{ f : X \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \le C \text{ a.e. on } X \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C; |f(x)| \le C \text{ a.e. on } X\}.$$

注记1.26.1. *If $f \in L^\infty$ then we have*

$$|f(x)| \le \|f\|_\infty \qquad \text{a.e. on } X.$$

Indeed, there exists a sequence C_n such that $C_n \to \|f\|_\infty$ and for each $n, |f(x)| \le C_n$ a.e. on X. Therefore $|f(x)| \le C_n$ for all $x \in X \backslash E_n$, with $|E_n| = 0$.

引理1.27 (Young's inequality)**.**

$$ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'} \quad \forall a \ge 0, \quad \forall b \ge 0.$$

Equivalently,

$$a^\lambda b^{1-\lambda} \le \lambda a + (1-\lambda) b, \quad \forall a \ge 0, \quad \forall b \ge 0, \quad \lambda \in (0,1),$$

with equality iff $a = b$.

定理1.28 (Hölder's inequality)**.** *Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \le p \le \infty$. Then $fg \in L^1$ and*

$$\int |fg| \le \|f\|_p \|g\|_{p'}.$$

注记1.28.1 (Extension of Hölder's inequality)**.** *Assume that $f_1, f_2, \ldots, f_k$ are functions such that*

$$f_i \in L^{p_i}, 1 \le i \le k \text{ with } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \le 1.$$

Then the product $f = f_1 f_2 \cdots f_k$ belongs to L^p and

$$\|f_1 f_2 \cdots f_k\|_p \le \|f_1\|_{p_1} \|f_2\|_{p_2} \cdots \|f_k\|_{p_k}.$$

In particular, if $f \in L^p \cap L^q$ with $1 \le p \le q \le \infty$, then $f \in L^r$ for all $r, p \le r \le q$, and the following "interpolation inequality" holds:

$$\|f\|_r \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}, \ \text{where} \ \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad 0 \le \alpha \le 1.$$

推论1.29. *If A is any set and $0 < p < q \le \infty$, then $l^p(A) \subset l^q(A)$ and $\|f\|_q \le \|f\|_p$.*

推论1.30. *If $\mu(X) < \infty$ and $0 < p < q \le \infty$, then $L^p(\mu) \supset L^q(\mu)$ and*

$$\|f\|_p \le \|f\|_q \mu(X)^{(1/p)-(1/q)}.$$

引理1.31 (Triangle inequality/ Minkowski inequality)**.** *For $1 \le p \le \infty$, and $f, g \in L^p$, we have*

$$\|f + g\|_p \le \|f\|_p + \|g\|_p. \tag{1.2}$$

In particular, L^p is a vector space and $\|\cdot\|_p$ is a norm for any $p, 1 \le p \le \infty$.

定理1.32. *For $1 \le p < \infty$, L^p is a Banach space.*

6.2 Reflexivity. Separability. Dual of L^p spaces

定义1.33 (Separable metric spaces)**.** *We say that a metric space E is separable if there exists a subset $D \subset E$, that is countable and dense.*

定义1.34. *The measurable space $(X, \mathcal{M})$ (or the σ-algebra $\mathcal{M}$) is called separable if there exists a countable family $\mathcal{C}$ such that $\sigma(\mathcal{C}) = \mathcal{M}$.*

定义1.35. *The measure space $(X, \mathcal{M}, \mu)$ is called μ-separable, if there exists a separable σ-algebra $\mathcal{M}_0 \subset \mathcal{M}$ such that $\forall A \in \mathcal{M}$, $\exists B \in \mathcal{M}_0$, $\mu(A \Delta B) = 0$.*

注记1.35.1. *$(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), m)$ is m-separable.*

定理1.36. *Assume that $(X, \mathcal{M}, \mu)$ is a μ-separable measure space, and μ is σ-finite. Then $L^p(X)$ is separable for any $p, 1 \le p < \infty$. Usually, $L^\infty(X)$ is not separable.*

作业P_{187} **7, 9, 10, 15;**