1 第8教学周10.24

6.5 Interpolation of L^p spaces

引理1.1 (The Three Lines Lemma). *Let ϕ be a bounded continuous function on the strip $0 \le \operatorname{Re} z \le 1$ that is holomorphic on the interior of the strip. If $|\phi(z)| \le M_0$ for $\operatorname{Re} z = 0$ and $|\phi(z)| \le M_1$ for $\operatorname{Re} z = 1$, then $|\phi(z)| \le M_0^{1-t} M_1^t$ for $\operatorname{Re} z = t$, $0 < t < 1$.*

定理1.2 (The Riesz-Thorin Interpolation Theorem). *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces and $p_0, p_1, q_0, q_1 \in [1, \infty]$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. For $0 < t < 1$, define p_t and q_t by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T is a linear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ into $L^{q_0}(\nu) + L^{q_1}(\nu)$ such that $\|Tf\|_{q_0} \le M_0 \|f\|_{p_0}$ for $f \in L^{p_0}(\mu)$ and $\|Tf\|_{q_1} \le M_1 \|f\|_{p_1}$ for $f \in L^{p_1}(\mu)$, then

$$\|Tf\|_{q_t} \le M_0^{1-t} M_1^t \|f\|_{p_t}$$

for $f \in L^{p_t}(\mu), 0 < t < 1$.

例1.3. *Fourier transform*

$$T(f)(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \xi} dx. \tag{1.1}$$

$$\|Tf\|_{L^\infty} \le \|f\|_{L^1}, \quad \|Tf\|_{L^2} = \|f\|_{L^2} \tag{1.2}$$

If $1 \le p \le 2$ and $1/p + 1/q = 1$, then the Fourier transform T has a unique extension to a bounded map from L^p to L^q, with $\|T(f)\|_{L^q} \le \|f\|_{L^p}$.

6.6 Convolution and regularization

Let $\Omega \subset \mathbb{R}^N$ be an open set.

$C(\Omega)$ is the space of continuous functions on Ω.

$C^k(\Omega)$ is the space of functions k times continuously differentiable on Ω($k \ge 1$ is an integer).

$C^\infty(\Omega) = \cap_k C^k(\Omega)$.

$C_c(\Omega)$ is the space of continuous functions on Ω with compact support in Ω, i.e., which vanish outside some compact set $K \subset \Omega$.

$C_c^k(\Omega) = C^k(\Omega) \cap C_c(\Omega)$.

$C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

If $f \in C^1(\Omega)$, its gradient is defined by

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_N}\right).$$

If $f \in C^k(\Omega)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index of length $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$, less than k, we write

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} f.$$

定理1.4 (Young). *Let $f \in L^1\left(\mathbb{R}^N\right)$ and let $g \in L^p\left(\mathbb{R}^N\right)$ with $1 \leq p \leq \infty$. Then for a.e. $x \in \mathbb{R}^N$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^N$ and we define*

$$(f \star g)(x) = \int_{\mathbb{R}^N} f(x-y) g(y) dy.$$

In addition $f \star g \in L^p\left(\mathbb{R}^N\right)$ and

$$\|f \star g\|_p \leq \|f\|_1 \|g\|_p$$

Let $f \in L^1\left(\mathbb{R}^N\right)$ and $g \in L^p\left(\mathbb{R}^N\right)$ with $1 \leq p \leq \infty$. Then

$$\operatorname{supp}(f \star g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

命题1.5. *Let $f \in C_c^k\left(\mathbb{R}^N\right)(k \geq 1)$ and let $g \in L_{\text{loc}}^1\left(\mathbb{R}^N\right)$. Then $f \star g \in C^k\left(\mathbb{R}^N\right)$ and*

$$D^\alpha(f \star g) = (D^\alpha f) \star g \quad \forall \alpha \text{ with } |\alpha| \leq k.$$

In particular, if $f \in C_c^\infty\left(\mathbb{R}^N\right)$ and $g \in L_{\text{loc}}^1\left(\mathbb{R}^N\right)$, then $f \star g \in C^\infty\left(\mathbb{R}^N\right)$.

定义1.6 (Mollifiers). *A sequence of mollifiers $(\rho_n)_{n \geq 1}$ is any sequence of functions on $\mathbb{R}^N$ such that*

$$\rho_n \in C_c^\infty\left(\mathbb{R}^N\right), \quad \operatorname{supp} \rho_n \subset \overline{B(0, 1/n)}, \quad \int \rho_n = 1, \rho_n \geq 0 \text{ on } \mathbb{R}^N.$$

例1.7.

$$\rho(x) = \begin{cases} e^{1/\left(|x|^2 - 1\right)} & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$\rho_n(x) = C n^N \rho(nx)$ with $C = 1/\int \rho$.

命题1.8. *Assume* $f \in C\left(\mathbb{R}^N\right)$. *Then* $(\rho_n \star f) \underset{n\to\infty}{\longrightarrow} f$ *uniformly on compact sets of* $\mathbb{R}^N$.

定理1.9 (density)**.** *The space* $C_c\left(\mathbb{R}^N\right)$ *is dense in* $L^p\left(\mathbb{R}^N\right)$*; i.e.,*

$$\forall f \in L^p\left(\mathbb{R}^N\right) \forall \varepsilon > 0 \quad \exists f_1 \in C_c\left(\mathbb{R}^N\right) \text{ such that } \|f - f_1\|_{L^p} \leq \varepsilon.$$

定理1.10. *Assume* $f \in L^p\left(\mathbb{R}^N\right)$ *with* $1 \leq p < \infty$. *Then* $(\rho_n \star f) \underset{n\to\infty}{\longrightarrow} f$ *in* $L^p\left(\mathbb{R}^N\right)$.

推论1.11. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. Then* $C_c^\infty(\Omega)$ *is dense in* $L^p(\Omega)$ *for any* $1 \leq p < \infty$.

推论1.12. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $u \in L^1_{\mathrm{loc}}(\Omega)$ *be such that*

$$\int uf = 0 \quad \forall f \in C_c^\infty(\Omega).$$

Then $u = 0$ *a.e. on* Ω.

6.7 Criterion for strong compactness in L^p

定理1.13 (Ascoli-Arzelà)**.** *Let* K *be a compact metric space and let* $\mathcal{F}$ *be a bounded subset of* $C(K)$. *The closure of* $\mathcal{F}$ *in* $C(K)$ *is compact if and only if* $\mathcal{F}$ *is uniformly equicontinuous, that is,*

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon \quad \forall f \in \mathcal{F}.$$

定理1.14 (Criterion for L^p strong compactness)**.** *Let* $\mathcal{F}$ *be a bounded set in* L^p *with* $1 \leq p < \infty$. $\mathcal{F}$ *is relatively compact if and only if*

1. $\lim_{h\to 0} \sup_{f\in\mathcal{F}} \|f(x+\hbar) - f(x)\|_{L^p} = 0$.
2. $\lim_{R\to\infty} \sup_{f\in\mathcal{F}} \int_{B_R^C} |f|^p dx = 0$.

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