1 第五教学周9.29

3.1 Signed measures

定义1.1 (Signed measure)**.** *Let $(X, \mathcal{M})$ be a measurable space. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that*

- $\nu(\emptyset) = 0$;
- ν *assumes at most one of the values* $\pm\infty$;
- *if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu(E_j)$, where the latter sum converges absolutely if $\nu\left(\bigcup_1^\infty E_j\right)$ is finite.*

命题1.2. *Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then*

$$\nu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then

$$\nu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \nu(E_j).$$

定义1.3. *If ν is a signed measure on $(X, \mathcal{M})$, a set $E \in \mathcal{M}$ is called positive (resp. negative, null) for ν if $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0, \nu(F) = 0$) for all $F \in \mathcal{M}$ such that $F \subset E$.*

引理1.4. *Any measurable subset of a positive set is positive, and the union of any countable family of positive sets is positive.*

定理1.5 (Hahn decomposition)**.** *If ν is a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \varnothing$. If P', N' is another such pair, then $P \Delta P' (= N \Delta N')$ is null for ν.*

引理1.6. *Let ν be a signed measure on $(X, \mathcal{M})$, and $A \in \mathcal{M}$ with $-\infty < \nu(A) < 0$. Then there is a negative set $B \subset A$ such that*

$$\nu(B) \leq \nu(A).$$

注记1.6.1. *The decomposition $X = P \cup N$ if X as the disjoint union of a positive set and a negative set is called a Hahn decomposition for ν. It is usually not unique (ν-null sets can be transferred from P to N or from N to P), but it leads to a canonical representation of ν as the difference of two positive measures.*

定义1.7. *We say that two signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular, or that ν is singular with respect to μ, or vice versa, if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing, E \cup F = X, E$ is null for μ, and F is null for ν.*

Informally speaking, mutual singularity means that μ and ν "live on disjoint sets." We express this relationship symbolically with the perpendicularity sign:

$$\mu \perp \nu.$$

定理1.8 (Jordan decomposition)**.** *If ν is a signed measure, there exist unique positive measures $\nu^+$and ν^-such that $\nu = \nu^+ - \nu^-$and $\nu^+ \perp \nu^-$.*

3.2 The Lebesgue-Radon-Nikodym theorem

定义1.9. *Suppose that ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ and write*

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.

注记1.9.1. *It is easily verified that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$*

定理1.10. *Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.*

推论1.11. *If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\int_E f d\mu\right| < \epsilon$ whenever $\mu(E) < \delta$.*

引理1.12. *Suppose that ν and μ are finite measures on $(X, \mathcal{M})$. Either $\nu \perp \mu$, or there exist $\epsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E (that is, E is a positive set for $\nu - \epsilon\mu$).*

定理1.13 (Lebesgue-Radon-Nikodym theorem)**.** *Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that*

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$, and any two such functions are equal μ-a.e.

作业P_{88} **3;** P_{92} **9, 11, 17;**