1 Measure Theory

1.1 Sets

The union and intersection of an arbitrary family $\mathscr{S}$ of subsets of a set X are defined by

$$\cup\mathscr{S} = \{x \in X : x \in S \text{ for some } S \text{ in } \mathscr{S}\}$$

and

$$\cap\mathscr{S} = \{x \in X : x \in S \text{ for each } S \text{ in } \mathscr{S}\}.$$

$$\begin{aligned}
&\text{(Commutative laws)} && A \cap B = B \cap A, \quad A \cup B = B \cup A, \\
&\text{(Distributive laws)} && A \cap \bigcup_k B_k = \bigcup_k (A \cap B_k), \quad A \cup \bigcap_k B_k = \bigcap_k (A \cup B_k), \\
& && A \cap \bigcap_k B_k = \bigcap_k (A \cap B_k), \quad A \cup \bigcup_k B_k = \bigcup_k (A \cup B_k), \\
&\text{(Associative laws)} && (A \cap B) \cap C = A \cap (B \cap C), \quad (A \cup B) \cup C = A \cup (B \cup C),
\end{aligned} \tag{1.1}$$

$$\text{(de Morgan's laws)} \quad \left(\bigcup_k A_k\right)^c = \bigcap_k A_k^c, \quad \left(\bigcap_k A_k\right)^c = \bigcup_k A_k^c. \tag{1.2}$$

They are valid for arbitrary (not necessarily countable) unions and intersections.

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k, \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k. \tag{1.3}$$

1.2 σ-algebra

定义1.1. *Let X be a nonempty set. Let $\mathcal{C}$ be a class of subsects of X.*

(1) *$\mathcal{C}$ is called a π-**system** if it is closed under finite intersections*

$$A, B \in \mathcal{C} \Longrightarrow A \cap B \in \mathcal{C}. \tag{1.4}$$

(2) *$\mathcal{C}$ is called a **semi-ring** if*

- *$\emptyset \in \mathcal{C}$.*
- *π-system.*

- $A, B \in \mathcal{C} \Longrightarrow A \setminus B \in \mathcal{C}_{\sum_f}$.

(3) $\mathcal{C}$ *is called a* ***ring*** *if*

- $\emptyset \in \mathcal{C}$.
- $A, B \in \mathcal{C} \Longrightarrow A \setminus B \in \mathcal{C}$.
- $A, B \in \mathcal{C} \Longrightarrow A \cup B \in \mathcal{C}$.

(4) $\mathcal{C}$ *is called a* ***semi-algebra*** *if*

- *semi-ring.*
- $X \in \mathcal{C}$.

(5) $\mathcal{C}$ *is called an* ***algebra*** *if*

- *it is closed under finite intersections.*
- *it is closed under complementation.* $A \in \mathcal{C} \Longrightarrow A^c \in \mathcal{C}$.

This implies that $(A \cap A^c = \emptyset)$

$$X \in \mathcal{C}, \quad \emptyset \in \mathcal{C}, \quad \text{finite unions}, \quad \text{difference operation}. \tag{1.5}$$

(6) $\mathcal{C}$ *is called a* σ***-algebra*** *if*

- *it is closed under coutable intersections.*
- *it is closed under complementation.* $A \in \mathcal{C} \Longrightarrow A^c \in \mathcal{C}$.

This implies that

$$X \in \mathcal{C}, \quad \emptyset \in \mathcal{C}, \quad \text{countable unions}, \quad \text{difference operation}. \tag{1.6}$$

There is a smallest σ*-algebra* $\{\emptyset, X\}$ *and a largest one* 2^X.

(7) $\mathcal{C}$ *is called a* ***monotone-class*** *if it is closed under monotone limits. That is, if* $A_n \in \mathcal{C}$ *with* $A_n \uparrow A$ *or* $A_n \downarrow A$*, then also* $A \in \mathcal{C}$.

(8) $\mathcal{C}$ *is called a* λ***-system*** *if*

(a) $X \in \mathcal{C}$.

(b) $A,\ B \in \mathcal{C},\ B \subset A \Longrightarrow A \setminus B \in \mathcal{C}$.

(c) $A_n \in \mathcal{C},\ n \geq 1,\ A_n \uparrow A \Longrightarrow A \in \mathcal{C}$.

注记1.1. (*a*) *and* (*b*) $\Longrightarrow$ *complementation* $\overset{(c)}{\Longrightarrow} A_n \in \mathcal{C} \downarrow A \Rightarrow A \in \mathcal{C}$.

注记1.2. *Let* $\left\{\mathcal{C}_i \in 2^X : i \in I\right\}$ *is a family of classes of subsets. If for every* $i \in I$, $\mathcal{C}_i$ *is closed under some operation, so is their intersection* $\bigcap_i C_i$.

定理1.2. *Let* $\mathcal{C} \subset 2^X$.

(1) *If* $\mathcal{C}$ *is closed under finite intersection, so is* $m(\mathcal{C})$.

(2) *If* $\mathcal{C}$ *is closed under finite intersection, so is* $\lambda(\mathcal{C})$.

(3) *If* $\mathcal{C}$ *is closed under complementation, so is* $m(\mathcal{C})$.

定理1.3 (Monotone classes). *Let* $\mathcal{C} \subset 2^X$.

(1) *If* $\mathcal{C}$ *is an algebra,* $m(\mathcal{C}) = \sigma(\mathcal{C})$.

(2) *If* $\mathcal{C}$ *is a* π*-system,* $\lambda(\mathcal{C}) = \sigma(\mathcal{C})$.

1.3 Measure

定义1.4. *A* ***measurable space*** *is a pari* $(X, \mathcal{F})$, *where* X *is a space and* $\mathcal{F}$ *is a* σ*-algebra in* X.

Given a measurable space $(X, \mathcal{F})$, *we say that a set function*

$$\mu : \mathcal{F} \mapsto \overline{R}_+ = [0, \infty], \tag{1.7}$$

is ***countably additive*** *if*

$$\mu(\sum_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n), \quad A_n \in \mathcal{F}, \quad A_n \cap A_m = \emptyset. \tag{1.8}$$

引理1.5 (continuity). *For any measure* μ *on* $(X, \mathcal{F})$ *and sets* $A_1, A_2, \ldots \in \mathcal{F}$, *we have*

(*i*) $A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$,

(*ii*) $A_n \downarrow A, \mu(A_1) < \infty \Rightarrow \mu(A_n) \downarrow \mu(A)$.

命题1.6. *Let* $\mathcal{C}$ *be a semi-ring. Assume that* $\mu : \mathcal{C} \mapsto \overline{\mathbb{R}}^+$, *and* $\mu(\emptyset) = 0$.

$$\mu \text{ is } \sigma\text{-additive} \iff \mu \text{ is finitely additive and semi-}\sigma\text{-additive.} \tag{1.9}$$

1.4 Outer measure

定义1.7 (Outer Measure). *Let* X *be a nonempty set. A nonnegative set function* $\mu^* : \mathcal{P}(X) \mapsto \overline{\mathbb{R}}^+$ *is called an outer measure if it satisfies*

- $\mu^*(\emptyset) = 0$,
- $\mu^*(A) \leq \mu^*(B),\ A \subset B$,
- $\mu^*(\bigcup_{j=1} A_j) \leq \sum_{j=1} \mu^*(A_j)$.

Let $\mathcal{C} \subset \mathcal{P}(X)$ and $\emptyset \in \mathcal{C}$. Assume that $\mu : \mathcal{C} \mapsto \overline{\mathbb{R}}^+$, and $\mu(\emptyset) = 0$.

Define

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu\left(A_n\right) : A_n \in \mathcal{C}, \quad A \subset \bigcup_{n=1}^{\infty} A_n \right\}, \tag{1.10}$$

or $+\infty$ if no such A_n exists, that is, $\inf \phi = +\infty$.

引理1.8 (Outer measure)**.** *μ^* is an outer measure induced by μ.*

Proof.

- subadditivity: Let $B_{j,i}$ be a $\mathcal{C}$-cover of A_j with

$$\sum_{i=1} \mu(B_{j,i}) \leq \mu^*(A_j) + \frac{\varepsilon}{2^j}. \tag{1.11}$$

Using the monotonicity, we have

$$\mu^*(\bigcup_{j=1} A_j) \leq \mu^*(\bigcup_{j=1} \bigcup_{i=1} B_{j,i}) \leq \sum_{j=1} \sum_{i=1} \mu(B_{j,i}) \tag{1.12}$$

$$\leq \sum_{j=1} \left(\mu^*(A_j) + \frac{\varepsilon}{2^j} \right) \leq \sum_{j=1} \mu^*(A_j) + \varepsilon. \tag{1.13}$$

□