1 第9教学周10.31

6.8 Criterion for weak compactness in L^p

定理1.1. *Let $1 < p \leq \infty$, and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^p(X)$ satisfying*

$$\sup_n \|f_n\|_{L^p} < \infty. \tag{1.1}$$

Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ and a function $f \in L^p(X)$ such that

$$\lim_{k\to\infty} \int f_{n_k} g d\mu = 0, \quad \forall g \in L^{p'}. \tag{1.2}$$

定理1.2 (Dunford-Pettis, weak compactness in L^1)**.** *Let $(X, \mathcal{M}, \mu)$ be a σ-finite and separable measure space. A set $\mathcal{F} \in L^1(X)$ is weakly sequentially compact if and only if*

- *$\mathcal{F}$ is bounded in $L^1(X)$.*
- *$\mathcal{F}$ is uniformly absolutely continuous.*
- *$\mathcal{F}$ is equi-tight.*

推论1.3. *Weak and strong convergence of sequences in l^1 are the same.*

6.9 The dual of L^∞

Let $(X, \mathcal{M}, \mu)$ be a measure space and let

$$F(X, \mathcal{M}, \mu) = \left\{ \nu : \mathcal{M} \mapsto \mathbb{C} \middle| \nu : \text{finitely additive set function}, \nu \ll \mu, \sup_{A\in\mathcal{M}|\nu(A)|<\infty} \right\}. \tag{1.3}$$

Define

$$|\nu|(A) = \sup \sum_{j=1}^{n} |\nu(A_j)| \tag{1.4}$$

where the supremum is taken over all disjoint decompositions $A = \sum_{j=1}^{n} A_j$.

Further, if $f \in L^\infty_\mu(X)$ and $\nu \in F(X)$,

$$\int_X f d\nu =: \lim_{n\to\infty}\int_X f_n d\nu$$

where f_n is simple, $\|f_n - f\|_\infty \to 0$, and

$$\int_X g d\nu = \sum_{j=1}^{n} a_j \nu(A_j) \ \text{ for } \quad g = \sum_{j=1}^{n} a_j 1_{A_j}.$$

i. $\forall f, g \in L_\mu^\infty(X), \forall \alpha, \beta \in \mathbb{C}$, and $\forall \nu \in F(X)$,

$$\int_X (\alpha f + \beta g) d\nu = \alpha \int_X f d\nu + \beta \int_X g d\nu$$

ii. $\forall f \in L_\mu^\infty(X)$ and $\forall \nu \in F(X)$, $\left|\int_X f d\nu\right| \le \int_X |f| d|\nu|$;

iii. $F(X)$ is normed by (5.36);

iv. If $f \in L_\mu^\infty(X), \nu \in F(X)$, and $\mu(\{x : f(x) \neq 0\}) = 0$, then

$$\int_X f d\nu = 0.$$

定理1.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space. There is a surjective isometric isomorphism*

$$\begin{aligned} F(X) &\to \left(L_\mu^\infty(X)\right)', \\ \nu &\mapsto F_\nu, \end{aligned}$$

where

$$\forall f \in L_\mu^\infty(X), \quad F_\nu(f) = \int_X f d\nu.$$

Also, $F(X)$ is a Banach space.

7.1 Positive linear functional on $C_c(X)$

X: locally compact Hausdorff (LCH) space.

定义1.5. *A linear functional I on $C_c(X)$ will be called positive if $I(f) \ge 0$ whenever $f \ge 0$.*

定义1.6 (regularity)**.** *Let μ be a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if*

$$\mu(E) = \inf\{\mu(U) : U \supset E, U \text{ open }\}$$

and inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact }\}.$$

定义1.7 (Radon measure)**.** *A Radon measure on X is a Borel measure that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets.*

定理1.8 (Riesz representation theorem on $C_c(X)$)**.** *If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that*

$$I(f) = \int f d\mu$$

for all $f \in C_c(X)$. Moreover, μ satisfies

$$\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\} \quad \text{for all open } U \subset X$$

and

$$\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\} \text{ for all compact } K \subset X.$$

作业

1. (Urysohn引理).设Ω为$\mathbb{R}^N$中开集，$K \subset \Omega$为紧集.证明存在$\psi \in C_c^\infty(\Omega)$满足

$$\psi(x) = 1, \quad \forall x \in K.$$

2. 有限Borel（复）符号测度空间$M(X, \mathcal{B}_X, \mathbb{R})$和$M(X, \mathcal{B}_X, \mathbb{C})$在变差范数下是Banach空间。

3. 设X是Banach空间。子集F相对紧$\Longleftrightarrow \forall \varepsilon > 0$,存在相对紧集$K_\varepsilon$,使得

$$F \subset K_\varepsilon + B(\varepsilon) := \{f + g : f \in K_\varepsilon, g \in B(\varepsilon)\}. \tag{1.5}$$

4. P_{192} **21;**