第三教学周9.19

定义0.1. *If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-measurable if*

$$\mu(D) \geq \mu^*(D \cap A) + \mu^*(D \cap A^c), \quad \forall D \subset X. \tag{0.1}$$

引理0.2 (Characteristic of μ^*-measurable set). *Let $\mathcal{C} \subset \mathcal{P}(X)$ and $\emptyset \in \mathcal{C}$. Assume that the non-negative set function $\mu : \mathcal{C} \mapsto [0, \infty]$ satisfies*

- $\mu(\emptyset) = 0$.
- *μ is semi-σ-additive.*

Let μ^* be the outer measure induced by μ.

The set $A \subset X$ is a μ^*-measurable set if and only if

$$\mu(C) \geq \mu^*(C \cap A) + \mu^*(C \cap A^c) \quad \forall C \in \mathcal{C}. \tag{0.2}$$

注记0.1. 验证可测集，减弱到只需要测试$\mathcal{C}$中元素即可。

定理0.3 (restriction of outer measure, Carathéodory). *Let μ^* be an outer measure on X, and $\mathcal{M}^*$ be the class of μ^*-measurable sets. Then $\mathcal{M}^*$ is a σ-algebra and the restriction of μ^* to $\mathcal{M}^*$ is a measure.*

Furthermore, $(X, \mathcal{M}^*, \mu^*\big|_{M^*})$ is a complete measure space.

Partial proof. Completeness: $\forall A \subset N$ with $\mu^*(N) = 0$,

$$\mu^*(D) \geq \mu^*(D \cap A^c) \geq \mu^*(D \cap A^c) + \mu^*(D \cap A), \quad \forall D \subset X. \tag{0.3}$$

□

定理0.4 (Uniqueness). *Conditions:*

- *$\mathcal{C}$: π-system, $X \in \mathcal{C}$.*
- *μ_1,μ_2: finite measure on $\sigma(\mathcal{C})$, $\mu_1|_{\mathcal{C}} = \mu_2|_{\mathcal{C}}$.*

Then $\mu_1 = \mu_2$ on $\sigma(\mathcal{C})$.

定理0.5 (Carathéodory extension's theorem). *Let $\mathcal{C}$ be a semi-ring of X, and a set function $\mu : \mathcal{C} \mapsto [0, \infty]$ be σ-additive. Then μ extends to a measure on $\sigma(\mathcal{C})$.*

(That is, $\mu^*\big|_{\sigma(\mathcal{C})}$, where μ^* is the outer measure induced by μ.)

Furthermore, if μ is σ-finite on $\mathcal{C}$, and $X \in \mathcal{C}_\sigma$, then the extension is unique.

定义0.6 (Complete measure space)**.** *Given the measure space* $(X, \mathcal{M}, \mu)$*, the* σ*-algebra* $\mathcal{M}$ *is called* μ*-complete if every subset of a* μ*-negligible set is measurable.*

定理0.7 (Completion)**.** *Let* $(X, \mathcal{M}, \mu)$ *be a measure space.*

$$\mathcal{N} = \{N \subset X | \exists A \in \mathcal{M}, \mu(A) = 0, \quad s.t. \quad N \subset A\} \tag{0.4}$$

$$\overline{\mathcal{M}} = \{A \cup N | A \in \mathcal{M}, N \in \mathcal{N}\} \tag{0.5}$$

$$\overline{\mu}(A \cup N) = \mu(A), A \in \mathcal{M}, N \in \mathcal{N} \tag{0.6}$$

Then $\overline{\mathcal{M}}$ *is a* σ*-algebra, and* $\overline{\mu}$ *is a measure on* $\overline{\mathcal{M}}$*.*

($\overline{\mu}$ *is a unique extension of* μ *to a complete measure on* $\overline{\mathcal{M}}$*.)*

$(X, \overline{\mathcal{M}}, \overline{\mu})$ *is the smallest complete measure space which contains* $(X, \mathcal{M}, \mu)$*. We call it the completion space of* $(X, \mathcal{M}, \mu)$*.*

定理0.8. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space,* μ^* *be the outer measure induced by* μ*. Then, for any* $E \subset X$*, there is a* $C \in \mathcal{M}$ *with* $E \subset C$ *and* $\mu^*(E) = \mu(C)$*.*

定理0.9. *Let* $(X, \mathcal{M}, \mu)$ *be a* σ*-finite measure space. Then* $\overline{\mathcal{M}} = \mathcal{M}^*$*.*

0.1 Borel/Lebesgue measures on Euclidean spaces

Let

$$\begin{aligned} &\mathcal{C} = \{(a, b] : a \leq b, a, b \in \mathbb{R}^n\}, \\ &\mu((a, b]) = \prod_{i=1}^{n} (b_i - a_i). \end{aligned} \tag{0.7}$$

引理0.10. $\mathcal{C}$ *is a semi-ring on* $\mathbb{R}^n$*, and* μ *is a* σ*-additive negative set function on* $\mathcal{C}$*.*

$\mathcal{B}(\mathbb{R}^n)$ is a Borel σ-algebra. $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^n)$.

定理0.11 (Lebesgue-Stieltjes measure)**.** *For any increasing and right continuous function* $F(x)$*, define*

$$\mu_F((a, b]) =: F(b) - F(a), \tag{0.8}$$

on the collection $\mathcal{C}$ *of intervals* $(a, b]$*. Then* μ_F *extends uniquely to a measure* μ_F^* *on* $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^n)$*.*

注记0.2. *When* $F(x) = x$*,* $\mathcal{L}(\mathbb{R}^n) =: \overline{\mathcal{B}(\mathbb{R}^n)}$ *is the collection of Lebesgue measurable sets.*

作业P_{32} **17, 23.**