1 第七教学周10.17

6.2 Reflexivity. Dual of L^p

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that p and q are conjugate exponents. Hölder's inequality shows that each $g \in L^q$ defines a bounded linear functional ϕ_g on L^p by

$$\phi_g(f) = \int fg$$

命题1.1. *Let $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $g \in L^q$, then*

$$\|g\|_q = \|\phi_g\| = \sup\left\{\left|\int fg\right| : \|f\|_p = 1\right\}.$$

If μ is semifinite, this result holds also for $q = \infty$.

命题1.2. *Let $1 < p < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. $L^q(X)$ is isometrically isomorphic to $(L^p(X))^*$. Furthermore, if μ is σ-finite, $L^\infty(X) = (L^1(X))^*$.*

The following table summarizes the main properties of the space $L^p(\Omega)$ when Ω is a measurable subset of $\mathbb{R}^d$:

	Reflexive	Separable	Dual space
L^p with $1 < p < \infty$	YES	YES	$L^{p'}$
L^1	NO	YES	L^∞
L^∞	NO	NO	Strictly bigger than L^1

6.3 Some useful inequalities

引理1.3 (Chebyshev's inequality)**.** *If $f \in L^p (0 < p < \infty)$, then for any $\alpha > 0$,*

$$\mu(\{x : |f(x)| > \alpha\}) \leq \left[\frac{\|f\|_p}{\alpha}\right]^p.$$

引理1.4 (Schur's lemma)**.** *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, and let K be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$. Suppose that there exists $C > 0$ such that $\int |K(x, y)| d\mu(x) \leq C$ for a.e. $y \in Y$ and $\int |K(x, y)| d\nu(y) \leq C$ for a.e. $x \in X$, and that $1 \leq p \leq \infty$. If $f \in L^p(\nu)$, the integral*

$$Tf(x) = \int K(x, y) f(y) d\nu(y)$$

converges absolutely for a.e. $x \in X$, the function Tf thus defined is in $L^p(\mu)$, and $\|Tf\|_p \leq C\|f\|_p$

定理1.5. *Let K be a Lebesgue measurable function on $(0,\infty)\times(0,\infty)$ such that*

$$K(\lambda x,\lambda y)=\lambda^{-1}K(x,y),\quad \text{for all } \lambda>0$$

and

$$\int_0^\infty |K(x,1)|x^{-1/p}dx=C<\infty \quad \text{for some } p\in[1,\infty],$$

and let q be the conjugate exponent to p. For $f\in L^p$ and $g\in L^q$, let

$$Tf(y)=\int_0^\infty K(x,y)f(x)dx,\quad Sg(x)=\int_0^\infty K(x,y)g(y)dy.$$

Then Tf and Sg are defined a.e., and

$$\|Tf\|_p\le C\|f\|_p \quad \text{and} \quad \|Sg\|_q\le C\|g\|_q.$$

推论1.6 (Hardy's inequality)**.** *Let*

$$Tf(y)=y^{-1}\int_0^y f(x)dx,\quad Sg(x)=\int_x^\infty y^{-1}g(y)dy.$$

Then for $1<p\le\infty$ and $1\le q<\infty$,

$$\|Tf\|_p\le\frac{p}{p-1}\|f\|_p,\quad \|Sg\|_q\le q\|g\|_q.$$

6.4 Distribution functions and weak L^p

定义1.7. *If f is a measurable function on $(X,\mathcal{M},\mu)$, we define its distribution function $\lambda_f:(0,\infty)\to[0,\infty]$ by*

$$\lambda_f(\alpha)=\mu(\{x:|f(x)|>\alpha\}).$$

命题1.8.

- *λ_f is decreasing and right continuous.*
- *If $|f|\le|g|$, then $\lambda_f\le\lambda_g$.*
- *If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.*
- *If $f=g+h$, then $\lambda_f(\alpha)\le\lambda_g\left(\frac{1}{2}\alpha\right)+\lambda_h\left(\frac{1}{2}\alpha\right)$.*

命题1.9. *If $\lambda_f(\alpha)<\infty$ for all $\alpha>0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then*

$$\int_X\phi\circ|f|d\mu=-\int_0^\infty\phi(\alpha)d\lambda_f(\alpha).$$

命题1.10. *If $0<p<\infty$, then*

$$\int|f|^pd\mu=p\int_0^\infty\alpha^{p-1}\lambda_f(\alpha)d\alpha.$$

定义1.11 (Weak L^p space). *If f is a measurable function on X and $0 < p < \infty$, we define*

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p}$$

or use the notations $\|f\|_{L^{p,w}}, \|f\|_{L^{p,\infty}}$. We define weak L^p to be the set of all f such that $[f]_p < \infty$.

注记1.12. *1. $[\cdot]_p$ is not a norm. The triangle inequality fails. However, weak L^p is a topological vector space.*

2. $L^p \subset weak - L^p$, and $[f]_p \leq \|f\|_{L^p}$.

3. Weak-L^∞=L^∞.

6.5 Interpolation of L^p spaces

Let T be a map from some vector space $\mathcal{D}$ of measurable functions on $(X, \mathcal{M}, \mu)$ to the space of all measurable functions on $(Y, \mathcal{N}, \nu)$.

- T is called sublinear if $|T(f+g)| \leq |Tf| + |Tg|$ and $|T(cf)| = c|Tf|$ for all $f, g \in \mathcal{D}$ and $c > 0$.
- A sublinear map T is strong type $(p,q)(1 \leq p, q \leq \infty)$ if $L^p(\mu) \subset \mathcal{D}, T$ maps $L^p(\mu)$ into $L^q(\nu)$, and there exists $C > 0$ such that $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p(\mu)$.
- A sublinear map T is weak type $(p,q)(1 \leq p \leq \infty, 1 \leq q < \infty)$ if $L^p(\mu) \subset \mathcal{D}, T$ maps $L^p(\mu)$ into weak $L^q(\nu)$, and there exists $C > 0$ such that $[Tf]_q \leq C\|f\|_p$ for all $f \in L^p(\mu)$. Also, we shall say that T is weak type (p, ∞) iff T is strong type (p, ∞).

定理1.13 (Marcinkiewicz interpolation theorem). *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces; p_0, p_1, q_0, q_1 are elements of $[1, \infty]$ such that $p_0 \leq q_0$, $p_1 \leq q_1$, and $q_0 \neq q_1$; and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}, \qquad \text{where } 0 < t < 1.$$

If T is a sublinear map from $L^{p_0}(\mu) + L^{p_1}(\mu)$ to the space of measurable functions on Y that is weak types (p_0, q_0) and (p_1, q_1), then T is strong type (p, q).

The Hardy-Littlewood maximal operator H:

$$Hf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| dy \quad \left(f \in L^1_{\text{loc}}(\mathbb{R}^n) \right).$$

推论1.14. *Let* $p \in (1, \infty)$*. There is a constant* $C(p) > 0$ *such that*

$$\|Hf\|_p \leq C(p)\|f\|_p, \quad \forall f \in L^p(\mathbb{R}^n).$$

作业P_{196} **27, 29;** P_{199} **36;** P_{208} **43, 45;**