第10讲：多元函数微分学习题课(I)

例1. 设方程：$u^3-3(x+y)u^2+z^3=0$ 确定了隐函数 $u=f(x,y,z)$，求 du.

解法(一)：对方程两边取全微分 d：

$$d(u^3-3(x+y)u^2+z^3)=d(0)=0 \Rightarrow d(u^3)-3d((x+y)u^2)+d(z^3)=0$$

$$\Rightarrow 3u^2du-3(dx+dy)u^2-3(x+y)2u\,du+3z^2dz=0$$，解得

$$du=\frac{3u^2dx+3u^2dy-3z^2dz}{3u^2-6u(x+y)}=\frac{u^2dx+u^2dy-z^2dz}{u^2-2u(x+y)}$$

解法(二)：令 $F(x,y,z,u)=u^3-3(x+y)u^2+z^3$，其中 x,y,z,u 地位相同，则 $F'_u=3u^2-6(x+y)u$；$F'_x=-3u^2$，$F'_y=-3u^2$，$F'_z=3z^2$.

且 $F'_u=3u^2-6(x+y)u\neq 0$，从而 $\frac{\partial u}{\partial x}=-F'_x/F'_u=+3u^2/(3u^2-6(x+y)u)$

$=\frac{u^2}{u^2-2(x+y)u}$；$\frac{\partial u}{\partial y}=-F'_y/F'_u=3u^2/(3u^2-6(x+y)u)=\frac{u^2}{u^2-2(x+y)u}$，$\frac{\partial u}{\partial z}=-\frac{F'_z}{F'_u}$

$=-\frac{3z^2}{3u^2-6(x+y)u}=\frac{-z^2}{u^2-2(x+y)u}$，于是

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=\frac{u^2dx+u^2dy-z^2dz}{u^2-2(x+y)u}$$

解法(三)：对方程两边分别对 x,y,z 求偏导，解出 $\frac{\partial u}{\partial x}$，$\frac{\partial u}{\partial y}$，$\frac{\partial u}{\partial z}$

再代入 $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$ 即可。 (1).

例2.(ex9.2/30, $u=u(x,y)\in C^2$)

试化方程：$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}\cdot 2+\frac{\partial u}{\partial y}\cdot 6=0$ 在

线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 下，可化简为：$\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$.

证法(一)：从线性变换 $\begin{cases}\xi=x+y\\ \eta=3x-y\end{cases}$ 可得线性逆变换：

$$\begin{cases}x=\frac{1}{4}(\xi+\eta)\\ y=\frac{1}{4}(3\xi-\eta)\end{cases}\Rightarrow \frac{\partial x}{\partial\xi}=\frac{1}{4}=\frac{\partial x}{\partial\eta};\ \frac{\partial y}{\partial\xi}=\frac{3}{4},\ \frac{\partial y}{\partial\eta}=-\frac{1}{4}.$$

从 $\frac{\partial u}{\partial\xi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\xi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\xi}=\frac{\partial u}{\partial x}\frac{1}{4}+\frac{\partial u}{\partial y}\frac{3}{4}\Rightarrow$

$$\frac{\partial^2 u}{\partial\eta\partial\xi}=\left(\frac{\partial u}{\partial\xi}\right)'_\eta=\left(\frac{1}{4}\frac{\partial u}{\partial x}\right)'_\eta+\left(\frac{3}{4}\frac{\partial u}{\partial y}\right)'_\eta=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial x\partial y}\frac{\partial y}{\partial\eta}\right)+$$

$$\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\cdot\frac{\partial x}{\partial\eta}+\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial\eta}\right)=\frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\frac{1}{4}+\frac{\partial^2 u}{\partial x\partial y}\left(-\frac{1}{4}\right)\right)+\frac{3}{4}\left(\frac{\partial^2 u}{\partial y\partial x}\frac{1}{4}+\frac{\partial^2 u}{\partial y^2}\left(-\frac{1}{4}\right)\right)$$

$$=\frac{1}{16}\left(\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}\right)$$

即 $$\begin{cases}\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=16\frac{\partial^2 u}{\partial\eta\partial\xi}\\ \frac{\partial u}{\partial x}+3\frac{\partial u}{\partial y}=4\frac{\partial u}{\partial\xi}\end{cases}$$

根据偏微分方程(PDE)化简为：$16\frac{\partial^2 u}{\partial\eta\partial\xi}+2\left(4\frac{\partial u}{\partial\xi}\right)=0$

即 $\frac{\partial^2 u}{\partial\eta\partial\xi}+\frac{1}{2}\frac{\partial u}{\partial\xi}=0$.

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证法(二)：从线性变换 $\begin{cases}\xi = x+y\\ \eta = 3x-y\end{cases} \Rightarrow \dfrac{\partial \xi}{\partial x}=1,\ \dfrac{\partial \xi}{\partial y}=1,$

$\dfrac{\partial \eta}{\partial x}=3,\ \dfrac{\partial \eta}{\partial y}=-1$. 而 $u(x,y)$ 通过中间变量可视为 ξ,η 的函数.

从而 $\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x}+\dfrac{\partial u}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x}=\dfrac{\partial u}{\partial \xi}\cdot 1+\dfrac{\partial u}{\partial \eta}\cdot 3.\\[2ex] \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y}+\dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y}=\dfrac{\partial u}{\partial \xi}\cdot 1+\dfrac{\partial u}{\partial \eta}(-1)\end{cases}$ ，从而

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}=\left(\dfrac{\partial u}{\partial \xi}\right)'_x+3\left(\dfrac{\partial u}{\partial \eta}\right)'_x=\dfrac{\partial^2 u}{\partial \xi^2}\cdot 1+\dfrac{\partial^2 u}{\partial \xi\partial \eta}\cdot 3+3\left(\dfrac{\partial^2 u}{\partial \eta\partial \xi}\cdot 1+\dfrac{\partial^2 u}{\partial \eta^2}\cdot 3\right)\\[2ex] \dfrac{\partial^2 u}{\partial x\partial y}=\left(\dfrac{\partial u}{\partial \xi}-\dfrac{\partial u}{\partial \eta}\right)'_x=\dfrac{\partial^2 u}{\partial \xi^2}\cdot 1+\dfrac{\partial^2 u}{\partial \xi\partial \eta}\cdot 3-\left(\dfrac{\partial^2 u}{\partial \eta\partial \xi}\cdot 1+\dfrac{\partial^2 u}{\partial \eta^2}\cdot 3\right)\\[2ex] \dfrac{\partial^2 u}{\partial y^2}=\left(\dfrac{\partial u}{\partial \xi}-\dfrac{\partial u}{\partial \eta}\right)'_y=\dfrac{\partial^2 u}{\partial \xi^2}\cdot 1+\dfrac{\partial^2 u}{\partial \xi\partial \eta}(-1)-\left(\dfrac{\partial^2 u}{\partial \eta\partial \xi}\cdot 1+\dfrac{\partial^2 u}{\partial \eta^2}(-1)\right)\end{cases}$$

即 $$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial \xi^2}+6\dfrac{\partial^2 u}{\partial \eta\partial \xi}+9\dfrac{\partial^2 u}{\partial \eta^2}\\[2ex] \dfrac{\partial^2 u}{\partial x\partial y}=\dfrac{\partial^2 u}{\partial \xi^2}+2\dfrac{\partial^2 u}{\partial \eta\partial \xi}-3\dfrac{\partial^2 u}{\partial \eta^2}\\[2ex] \dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial^2 u}{\partial \xi^2}-2\dfrac{\partial^2 u}{\partial \eta\partial \xi}+\dfrac{\partial^2 u}{\partial \eta^2}\end{cases}$$

且 $2\dfrac{\partial u}{\partial x}+6\dfrac{\partial u}{\partial y}=2\left(\dfrac{\partial u}{\partial \xi}+3\dfrac{\partial u}{\partial \eta}\right)+6\left(\dfrac{\partial u}{\partial \xi}-\dfrac{\partial u}{\partial \eta}\right)=8\dfrac{\partial u}{\partial \xi}$ ，从而

$\dfrac{\partial^2 u}{\partial x^2}+2\dfrac{\partial^2 u}{\partial x\partial y}-3\dfrac{\partial^2 u}{\partial y^2}+2\dfrac{\partial u}{\partial x}+6\dfrac{\partial u}{\partial y}=0$，化为

$\dfrac{\partial^2 u}{\partial \xi^2}(1+2-3)+\dfrac{\partial^2 u}{\partial \xi\partial \eta}\cdot(6+4+6)+\dfrac{\partial^2 u}{\partial \eta^2}(9-6-3)+8\dfrac{\partial u}{\partial \xi}=0$

即 $16\dfrac{\partial^2 u}{\partial \xi\partial \eta}+8\dfrac{\partial u}{\partial \xi}=0 \Rightarrow \dfrac{\partial^2 u}{\partial \eta\partial \xi}+\dfrac{1}{2}\dfrac{\partial u}{\partial \xi}=0.$ (3).

例3.(ex9.3/13):

设 $u=f(x,y,z)$，$\varphi(x^2,e^y,z)=0$，$y=\sin x$，且 $f,\varphi\in C^1$，$\frac{\partial\varphi}{\partial z}\neq 0$，求 $\frac{du}{dx}$。

解：(1). 从 $\varphi(x^2,e^{\sin x},z)=0$ 及 $\varphi_z'=\frac{\partial\varphi}{\partial z}\neq 0$，知：~~故由多~~

由方程 $\varphi(x^2,e^{\sin x},z)=0$ 可确定 z 是 x,y 的函数，从而

z 是 x 的复合函数。故从 $u=f(x,y,z)$ 知，u 是 x 的一元函数。

$$\frac{du}{dx}=f_1'\cdot 1+f_2'\cdot y_x'+f_3'\cdot z_x'=f_1'+f_2'\cdot\cos x+f_3'\cdot z_x'.$$

令 $F(x,y,z)=\varphi(x^2,e^y,z)$，则 $\begin{cases}F_x'(x,y,z)=\varphi_1'\cdot 2x+\varphi_2' e^{\sin x}\cos x\\ F_z'(x,y,z)=\varphi_3'\cdot 1=\varphi_3'.\end{cases}$

$$z_x'=-\frac{F_x'(x,y,z)}{F_z'(x,y,z)}=-\frac{2x\varphi_1'+e^{\sin x}\cos x\,\varphi_2'}{\varphi_3'}$$

故 $\frac{du}{dx}=f_1'\cdot 1+f_2'\cdot y_x'+f_3'\cdot\left(-\frac{2x\varphi_1'+e^{\sin x}\cos x\,\varphi_2'}{\varphi_3'}\right)$.

例4. 说明：全微分也具有一阶微分形式不变性。

即，若 $f(x,y)$ 可微，则不论 x,y 是自变量，还是中间变量，则 $z=f(x,y)$ 总有：

$$dz=df(x,y)=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=f'_x\cdot dx+f'_y\cdot dy \quad (*_1)$$

证(1)，当 x,y 是自变量且 $z=f(x,y)$ 可微分时，有

$$dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy,$$ 即 $(*_1)$ 成立.

(2) 当 $z=f(x,y)$ 可微分，且 $\begin{cases}x=g(s,t)\\ y=h(s,t)\end{cases}$ 可微分，且 $f(g(s,t),h(s,t))$ 有意义时，z 通过中间变量 x,y 成了变量 s,t 的函数，此时，

有：$dx=\frac{\partial x}{\partial s}ds+\frac{\partial x}{\partial t}dt$，$dy=\frac{\partial y}{\partial s}ds+\frac{\partial y}{\partial t}dt$. 且

$$dz=\frac{\partial z}{\partial s}ds+\frac{\partial z}{\partial t}dt=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)ds+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)dt$$

$$=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}ds+\frac{\partial x}{\partial t}dt\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}ds+\frac{\partial y}{\partial t}dt\right)$$

$$=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy,$$ 即 x,y 是中间变量时，$(*_1)$ 仍成立。

利用全微分的一阶微分形式不变性，可导出多元可微函数的如下的微分四则运算法则：

(1). $d(u\pm v)=du\pm dv$; (2). $d(u\cdot v)=vdu+udv$

(3) $d\left(\frac{u}{v}\right)=\frac{vdu-udv}{v^2}$，其中，$u,v$ 均可微，且 $v\neq 0$.

证(1). 令 $f(u,v)=u+v$，则 $f(u,v)\in C^1$，从而 $f(u,v)$ 可微。

无论 u,v 是自变量，还是中间变量，总有：

$$d(u+v)=df(u,v)=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=1du+1dv=du+dv$$

从而有：$d(u\pm v)=du\pm dv$. 这里 d 是全微分。

证(3). 令 $f(u,v)=\frac{u}{v}$，$(v\neq 0)$，则 $f\in C^1\Rightarrow d(\frac{u}{v})=df(u,v)$

$$=\frac{\partial f}{\partial u}du+\frac{\partial f}{\partial v}dv=\frac{1}{v}du+\left(\frac{-u}{v^2}\right)dv=\frac{vdu-udv}{v^2}.$$

设 $f(x,y)\in C^2$，则 $z=f(x,y)\Rightarrow dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$d(dz)\triangleq d^2z=d\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)=\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)'_x dx+\left(\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy\right)'_y dy$$

$$\xlongequal{dx,dy\text{视为常量}}\left(\frac{\partial^2 z}{\partial x^2}dx+\frac{\partial^2 z}{\partial x\partial y}dy\right)dx+\left(\frac{\partial^2 z}{\partial y\partial x}dx+\frac{\partial^2 z}{\partial y^2}dy\right)dy$$

$$=\frac{\partial^2 z}{\partial x^2}dx^2+\frac{\partial^2 z}{\partial x\partial y}dydx+\frac{\partial^2 z}{\partial y\partial x}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \quad (\text{※})$$

(※)是 x,y 是自变量时的 $z=f(x,y)$ 的二阶全微分. 二阶及以上的全微分通常不再有形式不变性。例. 设 $z=x^2+y^2$，x,y 是自变量

则 $dz=2xdx+2ydy$，$d^2z=2(dx)^2+2(dy)^2$，设 $\begin{cases} z=x^2+y^2 \\ x=s^2 \\ y=t^2 \end{cases}$，则 $z=s^4+t^4$，则 $\begin{cases} dx=2sds \\ dy=2tdt \end{cases}$

$d z=4s^3ds+4t^3dt$，$d^2z=12s^2(ds)^2+12t^2(dt)^2=3[4s^2(ds)^2+4t^2(dt)^2]=3((dx)^2+(dy)^2)$
$\neq 2((dx)^2+(dy)^2)$

例5.(ex9.3/11(3)).设$u=u(x,y)$, $v=v(x,y)$是由方程组:

$$\begin{cases} u=f(ux,v+y) \\ v=g(u-x,v^2y) \end{cases}$$ 所确定的隐函数组,求 变换 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

的Jacobi(雅可比)行列式 $\begin{vmatrix} u'_x, u'_y \\ v'_x & v'_y \end{vmatrix} \triangleq \frac{\partial(u,v)}{\partial(x,y)}$, $f,g\in C^1$.

解:令$A=ux$, $B=v+y$, $E=u-x$, $F=v^2y$,则$\begin{cases} u=f(A,B) \\ v=g(E,F). \end{cases}$

方程组两边关于x求偏导:$\begin{cases} u'_x=f'_1\cdot(u+xu'_x)+f'_2\cdot(v'_x+0) \\ v'_x=g'_1\cdot(u'_x-1)+g'_2\cdot 2vv'_xy \end{cases}$

标准化:$\begin{cases} (xf'_1-1)u'_x+f'_2\cdot v'_x=-uf'_1 \\ g'_1u'_x+(2vyg'_2-1)v'_x=g'_1 \end{cases}$

令$D=\begin{vmatrix} xf'_1-1, & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$.则$D\neq0$,且$D_1=\begin{vmatrix} -uf'_1 & f'_2 \\ g'_1 & 2vyg'_2-1 \end{vmatrix}$

$D_2=\begin{vmatrix} xf'_1-1 & -uf'_1 \\ g'_1 & g'_1 \end{vmatrix}$,由克莱姆(Cramer)法则

$u'_x=\frac{D_1}{D}$, $v'_x=\frac{D_2}{D}$.

方程组$\begin{cases} u=f(A,B) \\ v=g(E,F) \end{cases}$ 两边对y求偏导:

$$\begin{cases} u'_y=f'_1\cdot xu'_y+f'_2\cdot(v'_y+1) \\ v'_y=g'_1\cdot u'_y+g'_2\cdot(2vv'_yy+v^2) \end{cases} \Leftrightarrow \begin{cases} (xf'_1-1)u'_y+f'_2v'_y=-f'_2 \\ g'_1u'_y+(2vg'_2y-1)v'_y=-g'_2v^2. \end{cases}$$

(1).

$$D=\begin{vmatrix} xf_1'-1 & f_2' \\ g_1' & 2vyg_2'-1 \end{vmatrix}\neq 0,\ \tilde{D}_1=\begin{vmatrix} -f_2' & f_2' \\ g_2'v^2, & 2vyg_2'-1 \end{vmatrix},\ \tilde{D}_2=\begin{vmatrix} xf_1'-1 & -f_2' \\ g_1' & g_2'v^2 \end{vmatrix}$$

$$u_y'=\frac{\tilde{D}_1}{D},\quad v_y'=\frac{\tilde{D}_2}{D},\ \Rightarrow$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} u_x' & u_y' \\ v_x' & v_y' \end{vmatrix}=\begin{vmatrix} D_1 & \tilde{D}_1 \\ D_2 & \tilde{D}_2 \end{vmatrix}\Big/D.$$

例6 (ex9.3/15).

设 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$ 是由方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 确定的隐函数组，$F,G\in C^1$ 且 $\frac{\partial(F,G)}{\partial(u,v)}\neq 0$. 求 du,dv.

解: $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$; $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$

在 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 两边关于 x 求偏导得:

$$\begin{cases} F_x'\cdot 1+F_u'\cdot\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=0 \\ G_x'\cdot 1+G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=0 \end{cases},\ \text{整理得:}\begin{cases} F_u'\frac{\partial u}{\partial x}+F_v'\frac{\partial v}{\partial x}=-F_x' \\ G_u'\frac{\partial u}{\partial x}+G_v'\frac{\partial v}{\partial x}=-G_x' \end{cases}$$

令 $D=\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}$. 则 $D=\frac{\partial(F,G)}{\partial(u,v)}\neq 0$.

令 $D_1=\begin{vmatrix} -F_x' & F_v' \\ -G_x' & G_v' \end{vmatrix}$, $D_2=\begin{vmatrix} F_u' & -F_x' \\ G_u' & -G_x' \end{vmatrix}$, 则 $D_1=\begin{vmatrix} F_v' & F_x' \\ G_v' & G_x' \end{vmatrix}=\frac{\partial(F,G)}{\partial(v,x)}$

$D_2=\begin{vmatrix} F'_u & -F'_x \\ G'_u & -G'_x \end{vmatrix}=\begin{vmatrix} F'_x & F'_u \\ G'_x & G'_u \end{vmatrix}=\dfrac{\partial(F,G)}{\partial(x,u)}$，~~依cramner法则~~

$\dfrac{\partial u}{\partial x}=D_1/D=\dfrac{\partial(F,G)}{\partial(v,x)}\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$；$\dfrac{\partial v}{\partial x}=D_2/D=\dfrac{\partial(F,G)}{\partial(x,u)}\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$

对原方程组两边关于y求偏导. 由cramer法则，同样

可得：$\dfrac{\partial u}{\partial y}=\dfrac{\partial(F,G)}{\partial(v,y)}\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$；$\dfrac{\partial v}{\partial y}=\dfrac{\partial(F,G)}{\partial(y,u)}\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$.

于是，
$$\begin{cases} du=\dfrac{\partial u}{\partial x}dx+\dfrac{\partial u}{\partial y}dy=\left(\dfrac{\partial(F,G)}{\partial(v,x)}dx+\dfrac{\partial(F,G)}{\partial(v,y)}dy\right)\Big/\dfrac{\partial(F,G)}{\partial(u,v)}; \\ dv=\dfrac{\partial v}{\partial x}dx+\dfrac{\partial v}{\partial y}dy=\left(\dfrac{\partial(F,G)}{\partial(x,u)}dx+\dfrac{\partial(F,G)}{\partial(y,u)}dy\right)\Big/\dfrac{\partial(F,G)}{\partial(u,v)}. \end{cases}$$

例6 解法二：对原方程组两边同时取全微分：

$$\begin{cases} F'_x dx+F'_y dy+F'_u du+F'_v dv=d(0)=0 \\ G'_x dx+G'_y dy+G'_u du+G'_v dv=d(0)=0 \end{cases}$$，视du, dv为未知量，

依cramer法则，解得：$du=D_1/D=\left(\dfrac{\partial(F,G)}{\partial(v,x)}dx+\dfrac{\partial(F,G)}{\partial(v,y)}dy\right)\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$；

$dv=D_2/D=\left(\dfrac{\partial(F,G)}{\partial(x,u)}dx+\dfrac{\partial(F,G)}{\partial(y,u)}dy\right)\Big/\dfrac{\partial(F,G)}{\partial(u,v)}$. 其中，

$D_1=\begin{vmatrix} -(F'_x dx+F'_y dy) & F'_v \\ -(G'_x dx+G'_y dy) & G'_v \end{vmatrix}$，$D_2=\begin{vmatrix} F'_u & -(F'_x dx+F'_y dy) \\ G'_u & -(G'_x dx+G'_y dy) \end{vmatrix}$，$D=\dfrac{\partial(F,G)}{\partial(u,v)}$.

作业：ex9.2/31；ex9.3/6；7；8；10；11(1)；14.

附录：

例7. 设D是区域，$f\in C^2(D)$，$u=f(x+y+z, x^2+y^2+z^2)$，求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\partial y}$ 及 du.

解(1). 令 $v=x+y+z$，$w=x^2+y^2+z^2$，则 $u=f(v,w)$ 设 $\begin{cases} f_1'=\frac{\partial f}{\partial v}, \\ f_2'=\frac{\partial f}{\partial w}, \end{cases}$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial x}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}2x=f_1'+f_2'\cdot 2x;$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial w}\frac{\partial w}{\partial y}=\frac{\partial f}{\partial v}\cdot 1+\frac{\partial f}{\partial w}\cdot 2y=f_1'+f_2'\cdot 2y;$$

(2). $\frac{\partial^2 u}{\partial x^2}=(f_1'+2xf_2')'_x=(f_1')'_x+(2xf_2')'_x=f_{11}''\cdot 1+f_{12}''\cdot 2x+$
$(2x)'_x\cdot f_2'+2x(f_{21}''\cdot 1+f_{22}''\cdot 2x)=f_{11}''+4xf_{12}''+4x^2f_{22}''+2f_2'$.

$$\frac{\partial^2 u}{\partial x\partial y}=(f_1'+f_2'\cdot 2y)'_x=f_{11}''\cdot 1+f_{12}''\cdot 2x+2y(f_{21}''\cdot 1+f_{22}''\cdot 2x)$$
$$=f_{11}''+2f_{12}''\cdot(x+y)+4xyf_{22}''.$$

(3). $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$
$=(f_1'+2xf_2')dx+(f_1'\cdot+2yf_2')dy+(f_1'\cdot 1+2zf_2')dz$.

注：∵ $f\in C^2(D)$，∴ $f_{12}''=f_{21}''$，即 $\frac{\partial^2 f}{\partial w\partial v}=\frac{\partial^2 f}{\partial v\partial w}$.